Total Pages : 5 B.Sc./6th Sem/MTM/26(NEP)

2026

6th Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 13-T

(Single Core Major)

(Metric Space and Complex Analysis)

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : 2×10=20

1. State Banach fixed point theorem.

2. Prove that every convergent sequence is a Cauchy sequence in a metric space.

3. Prove that composition of two uniformly continuous functions is also uniformly continuous.

4. Let $\{x_n\}$ be sequence in X. If the sequence $\{x_n\}$ is convergent in the metric space (X,d), then for every particular $\alpha \in X$, the set $\{d(x_n, \alpha) : n \in N\}$ is bounded.

P.T.O.

5. Prove that every closed subset of a compact metric space is compact.

6. If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)[\operatorname{Re} f(z)]^2 = 2|f'(z)|^2.$$

7. Let C be the circle of radius 2 with centre at the origin in the complex plane, oriented in the anticlockwise direction. Then find the value of the integral $\oint_C \frac{dz}{(z-1)^2}$.

8. Define analytic function. Give an example of the function which satisfies Cauchy Riemann's equation but not differentiable.

9. Write necessary and sufficient condition for a function $f(z)$ to be differentiable.

10. Define the homomorphism of two metric spaces (X, d) and (Y, ρ). Give example.

11. Give an example of a complete metric space which is not compact.

12. Show that set of rational number $\mathbb{Q}$ is not complete.

13. In usual metric space R_n, prove that the set $\{x \in \mathbb{R} : |x| > 0\}$ is disconnected.

14. Find the analytic function f for which the real part is $e^{-z}\{(x^2+y^2)\cos y + 2xy\sin y\}$.

15. Without evaluating the integration show that $\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$ where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

Group - B

Answer any ***four*** questions : 5×4=20

16. Find the value of the integral $\int_0^{1+i}(x-y+ix^2)dz$ along with the real axis from $z = 0$ to 1 and then along with the line parallel to the imaginary axis from $z = 1$ to $1 + i$.

17. Show that the metric space $(C[a,b], d)$ is complete, where $d(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}$ for all $f, g \in C[a,b]$, $C[a,b]$ is the collection of all continuous functions on $[a, b]$.

18. Prove the Banach fixed point theorem.

19. Show that the continuous image of a connected set is connected.

20. Prove that if $f = u + iv$ is analytic in D then the C-R equation hold in D. If u and v have continuous first order partial derivatives in D and the C-R equation hold in D then f is analytic in D.

21. If z_1 is a point inside the circle of convergence $|z| = R$ of a power series $\sum_0^\infty a^n (z - z_0)^n$, then the series must be uniformly convergent in the closed disc $|z - z_0| \le R_1$ where $R_1 = |z_1 - z_0|$.

Group - C

Answer any ***two*** questions : 10×2=20

22. (a) Find the series expansion for $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the regions $1 < |z| < 4$.

(b) Let (X, d) be a metric space and $\{x_n\}$ is a sequence of elements of X converging to $x \in X$. Then prove that for any $\alpha \in X$, the sequence $\{d(x_n, \alpha)\}$ converges to $d(x, \alpha)$ in the real line.

5+5

23. (a) If $f(z)$ is an analytic function within and on a closed contour C and if a is any point within C, then show that $f(a) = \dfrac{1}{2\pi i} \int_C \dfrac{f(z)}{z-a} dz$.

(b) Let A be any non-empty subset of a metric space (X, d). The function $f : (X, d) \to R$ given by $f(x) = d(x, A)$ for all $x \in X$. Then show that $f(x)$ is continuous on the metric space. 5+5

24. (a) Let d be the standard discrete metric on the set of natural numbers $\mathbb{N}$ and d^* be the discrete metric on $\mathbb{N}$ given by $d^*(m,n) = \left|\dfrac{1}{m} - \dfrac{1}{n}\right|$, show that the metrics d and d^* are equivalent. Verify d and d^* are not uniformly equivalent.

(b) Let $f(z) = \sqrt{|xy|}$, show that $f'(0)$ doesn't exist but C-R equations are satisfied at the origin. 6+4

25. (a) State and proof Cauchy's integral formula.

(b) Let (X, d) be a complete metric space and $A \subset X$. Prove that $\bar{A}$ is compact $\Leftrightarrow$ A is totally bounded. 6+4
