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PG (CBCS)

M.SC. Semester- IV Examination, 2026

APPLIED MATHEMATICS

PAPER: MTM 401

(FUNCTIONAL ANALYSIS)

Full Marks: 40 **Time: 2 Hours**

The figures in the right-hand margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

GROUP-A

1. Answer any **FOUR** of the following questions: $4\times2=8$

a) State Hahn-Banach theorem.

b) Let H be a Hilbert space and F be a closed subspace of H. Prove that $H = F + F^{\perp}$.

c) If $T \in BL(H)$ is normal, prove that $Ker(T) = Ker(T^*)$.

d) Give an example of a normed space which is not an inner product space. Justify your answer.

e) Show that every normed space can be embedded as a dense subspace of a Banach space.

f) Suppose $S \in$ B(H), $T \in$ B(H), S and T are normal, and $ST = TS$. Prove that $S + T$ and ST are normal.

GROUP-B

2. Answer any **FOUR** of the following questions: $4\times4=16$

a) Let E be a measurable subset of $\mathbb{R}$ and for $t \in E$, let $x_1(t) = t$. Let $X = \{x \in L^2(E): x_1 x \in L^2(E)\}$ and $F: X \to L^2(E)$ be defined by $F(x) = x_1 x$. Show that if $E = [a, b]$, then F is continuous, but if $E = \mathbb{R}$, then F is not continuous.

b) For $1 \le p, q \le \infty$ and $x = (x_1, x_2 \dots, x_d)$ in $\mathbb{C}^d$, what are the best constants c and C such that $c\|x\|_p \le \|x\|_q \le C\|x\|_p$ for all $x \in \mathbb{C}^d$.

c) Show that if X is a locally compact normed space, then X is finite dimensional.

d) Suppose that X and Y are Banach spaces, $T: X \to Y$ is a linear map which has a closed graph. Prove that T is continuous.

e) Prove that every non-zero Hilbert space has an orthonormal basis.

f) Suppose X and Y are Banach spaces and $T \in B(X, Y)$ is injective. If T is bounded below then prove that $Rang(T)$ is closed.

(P.T.O)

GROUP-C

3. Answer any **TWO** of the following questions: 2×8=16

a) State and prove open mapping theorem. 2 + 6

b) Let, $\{u_1, u_2, u_3, ...\}$ be an orthogonal set in an inner product space X and let $k_1, k_2, k_3, ... \in \mathbb{C}$.

(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then prove that $k_n = \langle x, u_n \rangle$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$.

(ii)If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, prove that $\sum_{n=1}^{\infty} k_n u_n$ converges in X. 4+4

c)

(i) For $y \in l^{\infty}$, define $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j),\ \ x \in l^1$. Show that $\| f_y \| = \| y \|_{\infty}$ and the map $F: l^{\infty} \to (l^1)^*$ defined as $F(y) = f_y$ is onto.

(ii) A bounded monotone sequence of self-adjoint operators on a Hilbert space converges strongly. 5+3

d)

(i) Let X be compact and suppose that V is a Banach subspace of $C(X)$. If E is a closed subset of X such that for every $g \in C(E)$ there is an $f \in V$ with $f|E = g$, show that there is a constant $c > 0$ such that for each $g \in C(E)$ there is an $f \in V$ with $f|E = g$ and $\max\{|f(x)| : x \in X\} \leq c \max\{|g(x)| : x \in E\}$.

(ii) Show that $Ker(T) = Ran(T^*)^{\perp} = Ker(T^*T)$ if $T \in BL(H, K)$ and H, K are Hilbert spaces. 5+3

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