Total Pages : 5

B.Sc./5th Sem/MTM/25(NEP)

2025

5th Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 8-T (Single Core Major)

[Riemann Integration and Series of Functions]

Full Marks : 60

Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : 2×10=20

1. If f be a real-valued function on the closed interval $[a, b]$, define Riemann integral of f over $[a, b]$.

2. Prove that $\sup L(P, f) \le \inf U(P, f)$.

3. If f be Riemann integrable in $[a, b]$, prove that cf is also Riemann integrable in $[a, b]$, where c is a constant.

4. Prove that $\int_a^b -f(x)dx = -\int_a^b f(x)dx$.

5. State limiting definition of integration.

P.T.O.

6. Evaluate $\int_{-1}^{1} |3x+1| dx$.

7. How would you prove that $L(P, f) \le U(P_1, f)$, where P_1 is a refinement of P by distributing two additional points in P for a bounded function f in $[a, b]$?

8. Prove that if $f(x)$ be integrable in $[a, b]$, then $|f(x)|$ is also so.

9. If f and g be both bounded and integrable on $[a, b]$, and $f(x) \ge g(x)$, then prove that $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

10. Write two functions defined in a closed interval of which one has a primitive, but no integral and other has integral, but no primitive.

11. Define pointwise and uniform convergences of a sequence of functions.

12. Prove that $\Gamma(x+1) = x\Gamma(x), x > 0$.

13. When will a function be said to satisfy Dirichlet's conditions on an interval $[-\pi, \pi]$?

14. What do you mean by absolute and conditional convergences of a series of functions?

15. Prove that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$.

Group - B

Answer any ***four*** questions : 5×4=20

16. If $f(x)$ be integrable on $[a, b]$, prove that

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)| dx.$$

17. If $f(x)$ be bounded and integrable on $[a, b]$, prove that for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $0 \le U(P, f) - L(P, f) < \epsilon$.

18. If f be monotone increasing on $[a, b]$, prove that it is integrable on $[a, b]$.

19. Prove that $\{f_n(x)\}_n = \left\{\frac{x}{nx+1}\right\}_n$ converges uniformly to 0 on $0 \le x \le 1$.

20. Using definition of Riemann integral, evaluate $\int_0^1 x dx$ (taking n equal subintervals).

21. If f and g be integrable on $[a, b]$, prove that

$$\int_a^b (f+g)dx = \int_a^b f dx + \int_a^b g dx.$$

P.T.O.

Group - C

Answer any ***two*** questions : 10×2=20

22. (a) State and prove Cauchy condition for uniform convergence of a sequence of functions.

(b) Let

$f(x)=e^{-x}+2e^{-2x}+3e^{-3x}+\cdots\ldots+ne^{-nx}+\cdots,\ x>0.$

Show that $f(x)$ is continuous for all real $x>0$ and that the series can be integrated term-by-term over any finite interval $[a, b]$, where $a>0$. Prove also that $\int_{\log 2}^{\log 3} f(x)dx=\frac{1}{2}$. 6+4

23. (a) State and prove Darboux's theorem.

(b) Prove that $\frac{\sqrt{3}}{8}\le\int_{\pi/4}^{\pi/3}\frac{\sin x}{x}dx\le\frac{\sqrt{2}}{6}$. 6+4

24. (a) Discuss the convergence of Gamma function

$\Gamma(n)=\int_0^{\infty}e^{-x}x^{n-1}dx.$

(b) Show that the radius of convergence of $\sum a_n x^n$, where $a_{2n}=\frac{1}{3^n}$ and $a_{2n-1}=\frac{1}{3^{n+1}}$ is $\sqrt{3}$. 5+5

P.T.O.

25. (a) Obtain Fourier series corresponding to $f(x)=x$ on $[-\pi, \pi]$ and show that

$$x=2\left\{\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right\}$$ and

hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$.

(b) Show that for $k^2<1$,

$$\frac{\pi}{6}\le\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\le\frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{1}{4}k^2}}.$$ 5+5