B.Sc./3rd Sem/MTM/25(NEP)

2025

3rd Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 4-T (Single Core Major)

[Differential Equations and Vector Calculus]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any *ten* questions : $2\times10=20$

1. Define fundamental set of solutions for system of ordinary differential equation.

2. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solution of $x^2\frac{dy}{dx^2}-2x\frac{dy}{dx}$ for all x in $[0, 10]$ consider the Wronskian $W(x)=y_1(x)\frac{dy_2}{dx}-\frac{dy_1}{dx}y_2(x)$. If $W(1)=1$ then find the value of $W(3)-W(2)$.

P.T.O.

3. Show that $f(x, y) = xy^2$ on $\mathbb{R}:\{|x| \le 1, |y| \le 1\}$ satisfying a Lipschitz condition. But this function does not satisfy a Lipschitz condition on the strip $S:\{|x| \le 1, |y| \le \infty\}$.

4. If $y(x) = x^2 \sin x$ is a solution of an n-th order linear ODE

$$y^n(x) + a_1 y^{n-1}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0$$

with real constant coefficients, then prove that the least possible value of n is 6.

5. Evaluate $\int_0^1 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, where $\vec{r} = t^3\hat{\imath} + 2t^2\hat{\jmath} + 3t\hat{k}$.

6. If the volume of the tetrahedron is 2 cubic units and three of its vertices have position vectors (1, 1, 0), (1, 0, 1), (2, –1, 1), find the locus of the fourth vertex.

7. Find the formula for curvature (κ) and torsion (τ) for the curve : $x = x(t)$, $y = y(t)$, $z = 0$.

8. Show that the point at infinity is a regular singular point of the differential equation $x^2 \frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$.

9. Given that $\vec{r}(t) = \begin{cases} 2\hat{\imath} - \hat{\jmath} + 2\hat{k}, & \text{when } t = 2 \\ 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}, & \text{when } t = 3 \end{cases}$. Show that $\int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt} dt = 10$.

10. Given that $\vec{a} = 2\vec{l} - 3\vec{m} + 4\vec{n}$, $\vec{b} = \vec{l} + 2\vec{m} + 7\vec{n}$, $\vec{c} = 5\vec{l} - \vec{m} + 3\vec{n}$, where $\vec{l}$, $\vec{m}$, $\vec{n}$ are three non-coplanar vectors. Find $\vec{a} \cdot \vec{b} \times \vec{c}$.

11. Show that $y = x$ and $y = (x+1)^{-1}$ are linearly independent solutions of

$$(2x+1)(x+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = (2x+1)^2.$$

12. Examine the nature and stability of the critical point of the system $\frac{dx}{dt} = -4x - y$, $\frac{dy}{dt} = x - 2y$.

13. Find the volume of the parallelopiped whose edges are represented by $(3\hat{\imath} + 2\hat{\jmath} - 4\hat{k})$, $(3\hat{\imath} + \hat{\jmath} + 3\hat{k})$ and $(\hat{\imath} - 2\hat{\jmath} + \hat{k})$.

14. If $\vec{u} = t^2\hat{\imath} - t\hat{\jmath} + (2t+1)\hat{k}$, $\vec{v} = (2t-3)\hat{\imath} + \hat{\jmath} - t\hat{k}$ then find the value of $\frac{d}{dt}(\vec{u} \cdot \vec{v})$ at $t = 1$.

15. If $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$ then show that $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = \vec{0}$.

P.T.O.

Group - B

Answer any *four* questions : 5×4=20

16. Show that the solution of the ODE $\frac{dx}{dt}=2x+y$ and $\frac{dy}{dt}=3x$ satisfy the relation $3x+y=Ke^{3t}$, where K is a real constant.

17. Find a vector δ, which is perpendicular to both $\alpha=4\hat{i}+5\hat{j}-\hat{k}$ and $\beta=\hat{i}-4\hat{j}+5\hat{k}$ and which satisfies the relation $\delta.\gamma=21$ where $\gamma=3\hat{i}+\hat{j}-\hat{k}$.

18. If $\vec{r}=(2x^2y-x^4)\hat{i}+(e^{xy}-y\sin x)\hat{j}+(x^2\cos y)\hat{k}$, then find $\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2}$.

19. Solve the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x$ by the method of undetermined coefficients.

20. Show that the Frenet-Serret formulae can be written in the form :

$$\frac{d\hat{t}}{ds}=\vec{w}\times\hat{t},\quad \frac{d\hat{n}}{ds}=\vec{w}\times\hat{n},\quad \frac{d\hat{b}}{ds}=\vec{w}\times\hat{b};$$ determine $\vec{w}$.

21. Find the values of ρ for which all solutions of $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-\rho y=0$ tend to zero as $x\to\infty$.

Group - C

Answer any *two* questions : 10×2=20

22. (a) The functions $y_1, y_2, \dots y_n$ will be n linearly independent solutions of the equation $(D^n+P_1D^{n-1}+P_2D^{n-2}+\dots\dots\dots+P_n)y=F(x)$ on $[a, b]$ if and only if $W(y_1, y_2, \dots\dots y_n)(x)\neq 0$, $\forall x\in[a, b]$.

(b) Find the vector equation of the plane through the point $8\hat{i}+2\hat{j}-3\hat{k}$ and perpendicular to each of the planes $\vec{r}.(2\hat{i}-5\hat{j}+2\hat{k})=0$ and $\vec{r}.(\hat{i}+3\hat{j}-5\hat{k})+5=0$. 5+5

23. (a) Let r and q be the roots of the indicial polynomial for the equation $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$, where a, b are constants. If $r\neq q$, then show that two independent solutions are e^{rx} and e^{qx}.

(b) Prove that

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0.$$

6+4

P.T.O.

24. (a) If $\vec{A} = x^2yz\hat{\imath} - 2xz^3\hat{\jmath} + xz^2\hat{k}$, $\vec{B} = 2z\hat{\imath} + y\hat{\jmath} - x^2\hat{k}$;

find $\frac{\partial^2}{\partial x \partial y}\left(\vec{A} \times \vec{B}\right)$ at (1, 0, –2).

(b) Consider the system of linear differential equations :

$\frac{dx}{dt} = -3x + y$; $\frac{dy}{dt} = x - 3y$. Find the trajectories and phase curve in the phase plane of the system. Discuss the stability of the system. 4+6

25. (a) Solve the differential equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x$ by the method of variation of parameters.

(b) If $\vec{r} = \left(2x^2y - x^4\right)\hat{\imath} + \left(e^{xy} - y\sin x\right)\hat{\jmath} + \left(x^3\cos y\right)\hat{k}$

then find $\frac{\partial^2 \vec{r}}{\partial x^2} \times \frac{\partial^2 \vec{r}}{\partial y^2}$. 5+5
