B.Sc./5th Sem/MTM/25(NEP)

**2025**

# 5th Semester Examination (CCFUP : NEP)

## MATHEMATICS

**Paper : MJ 10-T (Single Core Major)**

**[Ring Theory and Linear Algebra-I]**



Full Marks : 60 Time : Three Hours

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*

### Group - A

Answer any ***ten*** questions : 2×10=20

1. Let $R$ be a ring and $a,b \in R$. Then prove that $(-a)b = a(-b) = -(ab)$.

2. Prove that a ring can have at most one multiplicative identity.

3. Does $\mathbb{Z}$ form a ring with respect to the usual addition '+' and '·' defined by $a \cdot b = \max\{a,b\}$? Justify your answer.

4. Give an example of a ring $R$ and its subring $S$ such that $S$ has multiplicative identity $1_s$ but $R$ does not have any.

P.T.O.

5. Consider the ring $M_2(\mathbb{Z}_3)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}_3\right\}$ w.r.t. usual addition and multiplication of matrices. Is $M_2(\mathbb{Z}_3)$ an integral domain? Justify your answer.

6. Find all maximal ideals of the ring $(\mathbb{Z}_6,+,\cdot)$.

7. Prove that $(\mathbb{Z},+,\cdot)$ is not isomorphic with $(2\mathbb{Z},+,\cdot)$ as rings.

8. Show that the mapping $f:\mathbb{Z}\left[\sqrt{2}\right] \to M_2(\mathbb{R})$ (where $M_2(\mathbb{R})$ denotes the ring of all 2 × 2 real matrices) defined by $f(a+b\sqrt{2})=\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ is a homomorphism of rings.

9. Examine whether $S=\{(x,y,z)|x+y+z=1\}$ forms a subspace of the real vector space $\mathbb{R}^2$.

10. Let $\{\alpha,\beta,\gamma)$ be a linearly independent set of vectors of a vector space $V$ over a field $F$. Then prove that for any $c \in F, \{\alpha+c\beta,\beta,\gamma\}$ is also linearly independent.

11. Let $T$ be the linear operator on the real vector space $\mathbb{R}^2$ defined by $T(a,b)=(-b,a)$ for all $(a,b)\in\mathbb{R}^2$. Find the matrix representation of $T$ with respect to the standard ordered basis of $\mathbb{R}^2$.



12. Let $U$ and $W$ be two subspaces of the real vector space $\mathbb{R}^3$ such that $B_U=\{(2,0,1),(3,1,0)\}$ is a basis for $U$ and $B_W=\{(1,0,0),(0,1,0)\}$ be a basis for $W$. Then find the dimension of $U\cap W$.

13. Prove that the intersection of two ideals is an ideal.

14. If the linear operator on $R^3$ defined by $T(a,b,c)=(0,a,b)$, then show that $T^3=0$, but $T\neq 0, T^2\neq 0$.

15. If $P(X)$ be the power set of a finite set $X$ of $n$ elements, then find the characteristic of the finite ring $(P(X),\Delta,\cap)$.

## Group - B

Answer any ***four*** questions : 5×4=20

16. Let $\omega\neq 1$ be a complex root of the equation $x^3-1=0$. Then prove that $T=\{a+b\omega|a,b\in\mathbb{Q}\}$ forms a subfield of the field of all complex numbers.

17. Let $V$ be a real vector space. Consider three subspaces $U_1,U_2,U_3$ of $V$ such that $U_1$ is spanned by $S_1=\{\alpha,\beta,\gamma\}$, $U_2$ is spanned by $S_2=\{\alpha,\alpha+\beta,\alpha+\beta+\gamma\}$ and $U_3$ is spanned by $S_3=\{\alpha+\beta,\beta+\gamma,\gamma+\alpha\}$. Then prove that $U_1=U_2=U_3$.

P.T.O.

18. Let $V$ be a vector space over a field $F$ and $\{a_1, a_2, a_3, \ldots, a_{2k}\}$ be a basis for $V$. Then prove that $\{a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots, a_{2k-1} + a_{2k}, a_{2k} + a_1\}$ need not form a basis for $V$ over $F$.

19. Let $V$ be a vector space over a field $F$ with $\dim V = n$. Then prove that $V$ is isomorphic with $F^n$ as vector spaces.

20. Prove that every finite integral domain is a field.

21. Prove that the cancellation laws hold in a ring if and only if it has no divisor of zero.

## Group - C

Answer any ***two*** questions : 10×2=20

22. (a) Does there exist an integral domain with 15 elements? Give justification in support of your answer.

(b) Find all ring homomorphisms from $(\mathbb{Z}_{12}, +, \cdot)$ to $(\mathbb{Z}_{30}, +, \cdot)$.

(c) Let $R$ and $S$ be two rings and $f : R \to S$ be an epimorphism. Show that if $I$ is an ideal of $R$ then $f(I)$ is an ideal of $S$. Give an example to show that the result is not true if $f$ is a homomorphism which is not surjective. 3+3+(2+2)





23. (a) Let $S_1$ ans $S_2$ be two non-empty subsets of a vector space $V$ over a field $F$. Then prove that $\text{Span}(S_1 \cup S_2) = \text{Span}\, S_1 + \text{Span}\, S_2$.

(b) Let $M_2(\mathbb{R})$ be the real vector space of all $2 \times 2$ real matrices. Then prove that $B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ forms a basis of $M_2(\mathbb{R})$.

(c) Describe explicitly, a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is the subspace spanned by (1, 0, –1) and (1, 2, 2). 3+4+3

24. (a) Let $\mathbb{Z}$ be the set of all integers. Prove that only homomorphisms from $\mathbb{Z} \to \mathbb{Z}$ are the identity and zero mappings.

(b) State and prove first isomorphism theorem of rings. 5+5

25. (a) Let $R$ be the ring $\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in Z \right\}$ and $\varphi : R \to Z$ is defined by $\varphi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b$.

Show that $\varphi$ is a ring homomorphism. Determine $\ker \varphi$. Show that the ring $R / \ker\varphi$ is isomorphic to $Z$.

(b) Show that the set of all real-valued continuous functions $y = f(x)$ satisfying the differential equation $\frac{d^3y}{dx^3} + 6\frac{d^3y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is a vector space over $R$. Find the basis of this space.

(3+1+2)+4