B.Sc./4th Sem (H)/MATH/23(CBCS)

2023

4th Semester Examination

MATHEMATICS (Honours)

Paper : GE 4-T

[CBCS]

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Numerical Methods]

Full Marks : 40 — Time : Two Hours

1. Answer any ***five*** of the following ; $2\times5=10$

   (a) Explain the principle of propagation of error and how it effects the numerical computation.

   (b) If $f(x) = 4\cos x - 6x$, find the relative percentage error in $f(x)$ for $x = 0$ if the error in x is 0.005.

   (c) What are the advantages and disadvantages for Lagrange interpolation method?

   (d) Compute the value of $\sqrt{2}$ correct up to three significant figure using Newton Raphson method.

   (e) If $f(1) = 3, f(2) = 7, f(3) = 13$ then find the value of $f'(1)$.

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(f) Find the value of the integral $\int_0^1 \frac{\ln(1+x)}{x} dx$ with step length 0.5 by Simpson's 1/3 rule.

(g) Show that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.

(h) Discuss the geometrical interpretation of trapezoidal rule for numerical integration.

2. Answer any ***four*** from the following : 5×4=20

(a) Obtain the function $f(x)$ as a polynomial in x using the following table :

x	0	2	4	6	8	10
$f(x)$	−1	5	10	17	29	49

(b) Compute $y(1.0)$ from $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$ taking $h = 0.2$, using Euler's method.

(c) Determine the largest (in magnitude) eigen value of the matrix given as follows using power method :

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}.$$

(d) Find the real root of $x^3 - x - 1 = 0$ using bisection method.

(e) Discuss Gauss-Seidal Iteration Scheme for solving the system of linear equations with the sufficient conditions of convergent.

(f) Describe the Newton-Raphson Method for finding the simple root of an equation $f(x) = 0$.

3. Answer any ***one*** of the following : 10×1=10

(a) Solve the following system of equations by LU decomposition method :

$2x - 3y + 4z = 8;\ x + y + 4z = 15;\ 3x + 4y - z = 8$

(b) Discuss the Newton's Backward interpolation formula and using it find a polynomial which takes the following values :

x	0	1	2	3	4	5
y	41	43	47	53	61	71

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OR

[Partial Differential Equations and Applications]

Full Marks : 60 Time : Three Hours

1. Answer any ***ten*** questions : 2×10=20

(a) Define central force and centre of force.

(b) Form a PDE by eliminating the function f from

$$z = xf\left(\frac{y}{x}\right).$$

(c) Prove the relation $pv = h$, the symbols have their usual meaning.

(d) A particle describes the curve $p^2 = ar$ under a force F to the pole. Find the law of force.

(e) A particle describes a curve $s = c\tan\psi$ with uniform speed v. Find the acceleration indicating its direction.

(f) Show that $u = f(x)\,g(x)$, where f and g are arbitrary twice differentiable functions, satisfies $u\,u_{xy} - u_x u_y = 0$.

(g) Write down the characteristic equation of the differential equation $u(x+y)u_x + u(x-y)u_y = x^2 + y^2$.

(h) Find the general solution of the PDE

$$4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0.$$

(i) Find a PDE of $\phi(u, v) = 0$ where $u = x + y + z$, $v = x^2 + y^2 + z^2$.

(j) Define quasilinear PDE and give an example of quasilinear PDE.

(k) Form the PDE by eliminating the arbitrary constants a and b from the equation $z = ax + a^2y^2 + b$.

(l) Classify the PDE $\left(1-K^2\right)\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,\ K > 1.$

(m) State Kepler's law for planetory motion.

(n) Write down the equations of motion of a particle moving along a smooth plane curve under the action of a force.

(o) Find the particular integral of

$$\left(D^2 - 2DD' + D'^2\right)z = \cos(x - 3y).$$

2. Answer any ***four*** questions : 5×4=20

(a) A particle moves in a plane with an acceleration always directed towards to a fixed-point O in the

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plane. Prove that the differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$, where $u=\frac{1}{r}$ and F is the acceleration.

(b) Find the integral surface of the linear partial differential equation $x\left(y^2+z\right)p-y\left(x^2+z\right)q=\left(x^2-y^2\right)z$ which contains the straight line $x+y=0,\ z=1$.

(c) Solve the differential equation

$z=px+qy+p^2+q^2$ by Charpits method.

(d) Transform the equation $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$ in equivalent canonical form and hence solve it.

(e) The temperature at one end of a bar 100 cm long with insulated sides is kept at 0°C and other end at 100°C until steady state condition prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

(f) A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.

3. Answer any ***two*** questions : 10×2=20

(a) Derive one dimensional wave equation and solve it by canonical reduction. 10

(b) Solve the heat conduction problem

$u_t=ku_{xx},\ 0<x<l,\ t>0$

$u(0,t)=0, \quad t\geq 0$

$u(l,t)=0, \quad t\geq 0$

$u(x,0)=f(x), \quad 0\leq x\leq l$ 10

(c) (i) Find the PDE of the family of cones whose vertex is the origin and base is the curve $x=a,\ \left(\frac{y}{a}\right)^2+z^2=b^2$ where a, b are arbitrary constants.

(ii) Prove that the partical differential equation $\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$ reduces to $\frac{\partial^2 z}{\partial u\partial v}=0$ by the transformation $u=x-ct,\ v=x+ct$. 6+4

(d) (i) Find the integral surface of the linear partial differential equation

$x\left(y^2+z\right)p-y\left(x^2+z\right)q=\left(x^2-y^2\right)z$

which contains the straight line $x+y=0$, $z=1$.

(ii) Solve : $\left(D^2+3DD'+2D'^2\right)z=2x^2+3y$. 6+4

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OR

[Ring Theory and Linear Algebra - I]

Full Marks : 60 Time : Three Hours

1. Answer any ***ten*** questions : 2×10=20

   (a) Give an example of a finite ring R with unity I_R and a subring S of R with unity I_S such that $I_R \neq I_S$.

   (b) Let $(R, +, \cdot)$ be a ring where the group $(R, +)$ is cyclic. Show that the given ring is commutative.

   (c) Let R be a ring having no zero divisors. Show that if every subring of R is an ideal of R then R is commutative.

   (d) Show that a non zero element 'a' in Z_n is a unit iff 'a' and 'n' are relatively prime. Also show that if 'a' is not a unit then it is a zero divisor.

   (e) Prove that in a ring R if 'a' is an idempotent element then $1 - a$ is also idempotent.

   (f) Examine whether $\{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 is a ring with unity.

   (g) Give an example (with reason) of a left ideal of a ring which is not a right ideal.

   (h) Define maximal ideal in a Ring. Give its example.

   (i) Find all the ring homomorphisms from $Z_{20} \to Z_{30}$.

   (j) Prove that Z and $Z \times Z$ are not isomorphic.

   (k) Show that the set $S = \{\sin x, \cos x, \sin(x+1)\}$ is linearly dependent.

   (l) Find the co-ordinate vector of the vector (2, 3, 3) with respect to the basis $B = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$.

   (m) Define rank and nullity of a linear transformation.

   (n) Is $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1+1, x_2+1, x_3+1)$ a linear transformation?

   (o) Find ker T of the linear transformation $T: R^3 \to R$ defined by $T(x, y, z) = x + y + z$, $(x, y, z) \in R^3$.

2. Answer any ***four*** questions : 5×4=20

   (a) Prove that a finite Integral Domain is a field. 5

   (b) Show that the ring of matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ contains no divisor of zero if $a, b \in \mathbb{Q}$ but contains divisor of zero if $a, b \in \mathbb{R}$. 5

   (c) Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and $\phi : R \to \mathbb{Z}$ is

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defined by $\phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) = a - b$. Show that ϕ is a ring homomorphism. Determine $\ker\phi$. Show that $R/\ker\phi \cong Z$. 2+1+2

(d) Prove that a vector space V can not be represented as the union of two proper subspaces. 5

(e) Find the basis and dimension of the subspace S of $\mathbb{R}^3$ defined by
$S = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$. 5

(f) Let $T : \mathbb{R}_{2\times2} \to \mathbb{R}_{2\times2}$ such that $T(A) = \frac{1}{2}(A + A^T)$, $A \in \mathbb{R}_{2\times2}$. Show that T is a linear mapping. Also find ker T, Im T, rank and nullity of T.

1+1+1+1+1

3. Answer any ***two*** questions : 10×2=20

(a) (i) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the order bases $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of T relative to the order bases $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3$ and $\{(1, 3), (2, 5)\}$ of $\mathbb{R}^2$. 5

(ii) A linear mapping $T : R^3 \to R^3$ is defined by $T(x, y, z) = (x - y, \quad x + 2y, \quad y + 3z)$, $(x, y, z) \in R^3$. Show that T is non-singular and determine T^{-1}. 5

(b) (i) Prove that any linearly independent set of n vectors of n dimensional vector space V is a basis of V. 4

(ii) Extend the set $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of R^4. 4

(iii) Prove that if W is a subspace of V then $L(W) = W$ and conversely. 2

(c) State and prove Fundamental theorem of Ring-homomorphism. Let $\phi : \mathbb{Z} \to \mathbb{Z}_5$ be defined by $\phi(x) = \bar{x} \pmod 5$ for $x \in \mathbb{Z}$. Show that ϕ is an onto homomorphism. Prove that $\mathbb{Z}/\ker\phi \simeq \mathbb{Z}_5$.

5+3+2

(d) (i) Let X be a non empty set. Then show that $P(X)$, the power set of X, forms a commutative ring with unity under + and · defined by $A + B = (A \cup B) - (A \cap B)$, $A \cdot B = A \cap B$. 5

(ii) Prove that in a commutative ring with unity, every maximal ideal is a prime ideal. Is the converse true? Justify. 5

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OR

[Multivariate Calculus]

Full Marks : 60 Time : Three Hours

1. Answer any ***ten*** questions : 2×10=20

(a) The function $f(x, y) = 3x + 12y - x^3 - y^3$ has two saddle points. Is the statement true? Justify.

(b) Find the surface area of the part of the surface $z = x^2 + y^2$ below the plane $z = 9$.

(c) Find the following limit, if it exists, or show that the limit does not exist.

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - xy + y^2}{x^2 + y^2}$$

(d) Find the directional derivative of the function $f(x, y, z) = xyz$ in the direction of vector $v = < 5, -3, 2 >$.

(e) Find the Jacobian of the transformation $x = u^2 - v^2$, $y = uv$.

(f) Suppose the vector field $F = yi + (x + z)j + (y + 2z)k$ is conservative. Find a potential function of F.

(g) An object moves along the line segment from

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(0, 0, 0) to (3, 6, 10), subject to the force $F = (x^2, y^2, z^2)$. Find the work done.

(h) State and prove the Fundamental Theorem of Calculus for Line Integrals.

(i) Find a double integral equal to the volume of the solid bounded by the surfaces $y = x$, $x = 2$, $z = 0$ and $z = y$. Note that you do not need to evaluate it.

(j) What is a homogeneous function of two variables? Give an example with justification.

(k) State sufficient condition for differentiability of a function $f(x, y)$ at a point (a, b).

(l) Evaluate $\int_0^{\pi} \int_0^{x} \sin y \, dy \, dx$.

(m) For what values of x the vector field $\vec{F} = \left(x^2\hat{i} + 2y\hat{j} + 3z\hat{k}\right)$ is solenoidal?

(n) State the Gauss's Divergence theorem.

(o) Find the equation of the tangent plane to the surface $xyz = 4$ at the point (1, 2, 2).

2. Answer any ***four*** questions : 5×4=20

(a) Find the tangent plane and normal line to

$$e^{xy^2} + zy^4 = 60 + \frac{z^2}{x+1} \text{ at } (0, -2, 8).$$

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(b) Find the volume enclosed between a sphere of radius 'a' centred on the origin, and a circular cone of half angle α with its vertex at the origin.

(c) Show that the necessary and sufficient condition for a proper vector $\vec{u}$ has a constant length is that $\vec{u}.\frac{d\vec{u}}{dx}=0$.

(d) State Euler's theorem on homogeneous function in two variables and use it to show that if $u=\tan^{-1}\frac{x^3+y^3}{x-y}$ then prove that $x\frac{\partial u}{dx}+y\frac{\partial u}{dy}=\sin 2u$.

(e) Changing the order of integration, show that

$$\int_0^1 dx\int_x^{\frac{1}{x}}\frac{ydy}{(1+xy)^2(1+y^2)}=\frac{\pi-1}{4}.$$

(f) Find the total work done in moving a particle in a force field given by

$$\vec{F}=(2x-y+z)\hat{i}+(x+y-z)\hat{j}+(3x-2y-5z)\hat{k},$$

along a circle C in the xy-plane $x^2+y^2=9$, $z=0$.

3. Answer any ***two*** questions : 10×2=20

(a) (i) Use Lagrange Multipliers to find the maximum

and minimum values of $z=f(x,y)=4x+6y$ subject to the constant $x^2+y^2=2023$.

(ii) For the vector field that is conservative, evaluate the line integral $\int_C \vec{F}.\vec{dr}$, where C is any curve from (0, 0) to (0, 1). 7+3

(b) (i) Your steel company produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $S(x,y,z)=8xyz^2-200(x+y+z)$. The company is to produce 100 units annually. How should production be distributed to maximize revenue?

(ii) State the Green's theorem and use it to evaluate the line integral $\int_C(1+xy^2)dx-x^2ydy$ where C consists of the arc of the parabola $y=x^2$ from (– 1, 1) to (1, 1). 5+5

(c) (i) Verify the divergence theorem for $\iint_S x^2z^2dS$, where S be the surface of the sphere $x^2+y^2+z^2=2023^2$.

(ii) Show that the following integral is independent of the path $\int_{(1,0)}^{(3,2)}\left[x+2ydx+(2x-y)dy\right]$. 6+4

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(d) (i) Given that z is a function of x and y and that $x = u + v$, $y = u\,v$, prove that

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = \left(x^2 - 4y\right)\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}.$$

(ii) Evaluate $\iiint 2x\, dv$ over the region $2x + 3y + z = 6$ that lies in the first octant.

6+4

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