বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2023

(Under CBCS Pattern)

Semester — II

Subject : MATHEMATICS

Paper : GE–2T

(Algebra)

Full Marks : 60

Time : 3 hours

The figures in the right-hand margin indicate marks.

The symbols used have their usual meanings.

Answer from **all** the Groups as directed.

GROUP—A

1. Answer *any* **ten** questions from the following : $2\times10=20$

   (a) State and prove the triangle inequality of complex numbers.

(b) Find the product of all the values of $(1+i)^{\frac{4}{5}}$.

(c) Apply Descartes' rule of signs to find the nature of the roots $x^7 + x^5 - x^3 = 0$.

(d) If a, b, x and y all are positive real numbers, then prove that

$$\frac{ax+by}{a+b} \geq \frac{(a+b)xy}{ay+bx}$$

(e) Find the number of multiple roots of the equation

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

(f) Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 3 \\ 1 & 7 & 2 & 1 \end{pmatrix}$.

(g) Is the union of two subspaces of a vector space V form a subspace of V? Justify.

(h) Find the remainder when $1! + 2! + 3! + \cdots + 80!$ is divided by 30.

(i) Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } y = 0\}$. Is W a subspace of $\mathbb{R}^3$? Why?

(j) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(x, y) = (2x - y, -8x + 4y)$. Does $(5, 10)$ belong to $\ker(T)$? Justify.

(k) Using the principle of induction, prove that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer for all $n \in \mathbb{N}$.

(Continued)

(l) Let α, β, γ, δ be the roots of the equation $x^4 + qx^2 + rx + s = 0$. Find $\sum \frac{1}{\alpha^2}$.

(m) Find the values of k for which the system of equations

$$\begin{cases} x + y + z = kx \\ x + y + z = ky \\ x + y + z = kz \end{cases}$$

will have non-trivial solution.

(n) Let A be a 7×6 matrix such that $Ax = 0$ has only the trivial solution.

(o) Show that the map $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = x + y + z$, $\forall x, y, z \in \mathbb{R}$ is a linear map.

GROUP—B

2. Answer *any* **four** questions from the following : 5×4=20

(a) Solve the equation by Ferrari's method $x^4 - 2x^2 + 8x - 3 = 0$.

(b) If a, b, c are positive rational numbers, then show that

$$a^a b^b c^c \geq \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \left(\frac{b+c}{2}\right)^{\frac{b+c}{2}} \left(\frac{c+a}{2}\right)^{\frac{c+a}{2}} \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

(Turn Over)

(c) If a, b, c are the roots of the equation $x^3 + qx + r = 0$, then find the equation whose roots are $(a-b)^2, (b-c)^2, (a-c)^2$.

(d) If a, b, c are positive real numbers such that $a+b+c=1$, then show that

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} < 5 .$$

(e) State Cayley-Hamilton theorem for matrix.

Using it, compute the inverse of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$.

1+4=5

(f) Find a basis and dimension of the subspace $S \cap T$ of $\mathbb{R}^4$ where $S = \{(x,y,z,w) \in \mathbb{R}^4 \mid x+y+z+w=0\}$ and $T = \{(x,y,z,w) \in \mathbb{R}^4 \mid 2x+y-z+w=0\}$

GROUP—C

3. Answer *any* **two** questions from the following : 10×2=20

(a) (i) Let $f(x) = x^4 + 6x^2 + 14x^2 + 22x + 5$. Find α, β and λ so that $f(x)$ may be expressed in the form $(x^2+3x+\lambda)^2 - (\alpha x + \beta)^2$. Hence solve the equation $f(x) = 0$.

(ii) If $\gcd(a,b)=1$, then show that $\gcd(a+b, a^2-ab+b^2) = 1$ or 3.

(3+3)+4=10

(b) (i) Define orthogonal matrix. Prove that if λ be an eigenvalue of a real orthogonal matrix A, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

(ii) Let A be a 3×3 real matrix with its eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ corresponding to eigenvalues 2, 3 and 1 respectively. Find A. (1+4)+5=10

(c) (i) Let $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ be the order bases of the real vector space V and W respectively. A linear mapping $T: V \to W$ maps the basis vectors as $T(\alpha_1) = \beta_1 + \beta_2$, $T(\alpha_2) = \beta_2 + \beta_3$, $T(\alpha_3) = \beta_3$. Find the matrix of T relative to the order bases $(\alpha_1, \alpha_2, \alpha_3)$ of V and $(\beta_1, \beta_2, \beta_3)$ of W. Deduce that T is invertible. Find the matrix of T^{-1} relative to the same chosen order bases.

(ii) Find the dimension of the subspace S of R^3 defined by

$$S = \{(x, y, z) \in R^3 : x + 2y = z, \; 2x - y + 3z = 0\}$$

6+4=10

(d) (i) Let R_1 be a relation defined on the set of integers $\mathbb{Z}$ such that

$$R_1 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y = 5n\}$$

Show that R_1 is an equivalence relation.

(ii) Find the roots of the equation $z^n = (z+1)^n$, where z is a complex number and n is a positive integer.

(iii) Let V be a vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Verify whether the transformation $T : V \to \mathbb{R}^2$ given by $T(f) = (f(0), f(1) + 1)$ is linear.

4+3+3=10

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