B.Sc./5th Sem (H)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 2-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Probability and Statistics]

1. Answer any ***ten*** of the following : $2\times10=20$

(a) In a game of bridge, the entire deck of 52 cards is dealt out to 4 people. Find the probability that one of the players receive all 13 clubs.

(b) In a bolt factory, machines A, B, C manufactures respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the production and is found defective. What is the probability that it was produced by the machine A?

P.T.O.

(c) The probability density function of a random variable X is given by $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1\le x<2 \\ 0, & \text{elsewhere.} \end{cases}$

Find the value of $P\left(\frac{1}{2}<X<\frac{3}{2}\right)$.

(d) Find the moment generating function of a random variable X uniformly distributed over $(-1, 1)$.

(e) Show that the probability that two numbers, chosen at random, will be prime to each other is $6/\pi^2$.

(f) Find the value of K for which

$$f(x, y)=\begin{cases} K\left(1-x^2-y^2\right), & 0<x^2+y^2<1 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability density function of a two dimensional random variable (X, Y).

(g) Show that correlation coefficient between two random variables is numerically 1, if and only if they are linear.

(h) Do you consider these two lines $2x+3y=7$, $3y-7x+2=0$ as the regression lines? Give reason.

(i) A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.

(j) Two random variables X, Y have the least square regression lines with equations $3x+2y-26=0$ and $6x+y-31=0$. Find the correlation coefficient between them.

(k) State Tchebycheff's inequality and give the physical significance of it.

(l) A coin is tossed 4 times and p is the probability of getting head in a single trial. Let s be the number of head(s) obtained. It is decided to test $H_0: p=\frac{1}{2}$ against $H_1: p\neq\frac{1}{2}$, using the decision rule. Reject H_0 if s is 0 or 4. Then find the type-I error.

(m) Find the value of k so that the following table may represent a joint distribution

	$Y=1$	$Y=2$
$X=1$	0.4	0.1
$X=2$	k	0.3

(n) If the mean and standard deviation of 10 observations $x_1, x_2, ..., x_{10}$ are 2 and 3 respectively, then find the mean of $(x_1+1)^2, (x_2+1)^2, \cdots, (x_{10}+1)^2$.

P.T.O.

(o) Let X, Y, Z be three random variables, with $\sigma_x = 2$, $\sigma_y = 1$ and $\sigma_z = 3$; $\rho_{xy} = 0.3$, $\rho_{yz} = 0.5$ and $\rho_{zx} = 0.5$. Find the variance of $U = X + Y - Z$.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Two points are taken at random on the given straight line of length a. Prove that the probability of their distance exceeding a given length c $(< a)$ is $\left(1-\frac{c}{a}\right)^2$.

(b) If $f(x,y) = 3x^2 - 8xy + 6y^2$ $(0 < x < 1,\ 0 < y < 1)$, then find $f_x(x/y)$ and $f_y(y/x)$. Hence or otherwise show that X, Y are dependent.

(c) If a random variable X possesses a finite second order moment about $c \in \mathbb{R}$, then show that, for any $\epsilon > 0$,

$$P(|X - c| \geq \epsilon) \leq \frac{E\{(X-c)^2\}}{\epsilon^2}.$$

(d) The discrete random variable X has the power series distribution with probability mass function

$f(x) = a_x \cdot \frac{\theta^x}{g(\theta)}$ for $x = 0, 1, 2, \ldots$, where $g(\theta)$

is a differentiable function of the parameter θ. Show that $E(X) = \theta \frac{d}{d\theta}\{\log g(\theta)\}$ and

$$\operatorname{var}(X) = E(X) + \theta^2 \frac{d^2}{d\theta^2}\{\log g(\theta)\}.$$ 2+3

(e) If X is a $N(m, \sigma)$ variate, then prove that $\mu_{2K+2} = \sigma^2 \mu_{2K} + \sigma^3 \frac{d\mu_{2K}}{d\sigma}$, where μ_K denotes the K-th central moment. Hence find μ_4.

(f) Prove that the maximum likelihood estimate of the parameter α of the population having density function $f(x) = \frac{2(\alpha - x)}{3\alpha^2}$, $0 < x < \alpha$ for a sample x_1 of unit size is $2x_1$. Test whether the estimate is biased or not.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then one ball is taken at random from the second urn. What is the probability that the ball drawn is a white ball? 4

P.T.O.

(ii) The joint probability density function of a two dimensional random variables (X, Y) is given by

$$f(x,y) = e^{-(x+y)}; \quad 0 \le x < \infty, \; 0 \le y < \infty.$$

Find (I) $P(X < Y \mid X < 2Y)$,

(II) $P(1 < X + Y < 2)$. 3+3

(b) (i) Let the random variable X has the marginal density $f_1(x) = 1, \; -\frac{1}{2} < x < \frac{1}{2}$ and the conditional density of Y is

$$f(y/x) = \begin{cases} 1, & x < y < x+1, \quad -\frac{1}{2} < x < 0 \\ 1, & -x < y < 1-x, \quad 0 < x < \frac{1}{2}. \end{cases}$$

Show that X and Y are uncorrelated. 5

(ii) State limit theorem for characteristic function. Use it to obtain Poisson distribution as a limiting case of binomial distribution. 5

(c) (i) Suppose (X, Y) is a two dimensional random variable. Show that $\{E(X,Y)\}^2 \le E(X^2)E(Y^2)$. Deduce that $-1 \le \rho(X,Y) \le 1$, where ρ is the correlation coefficient between X and Y. 2+3

(ii) Nine patients to whom a certain drug was

administered, registered the following rise in blood pressure in mm of Hg : 3, 7, 4, –1, –3, 6, –4, 1, 5. Test the hypothesis that the drug did not raise blood pressure at 10% significance level assuming that the sample is chosen from a normal population. Given : $P(t > 1.86) = 0.05$ for 8 degrees of freedom. 5

(d) (i) Obtain 99% confidence interval of the population variance σ^2 on the basis of the data $\Sigma_{i=1}^{10} x_i = 620$ and $\Sigma_{i=1}^{10} x_i^2 = 39016$. Note that $X_1, \ldots, X_{10}$ follows normal distribution. It is given that $\chi^2_{0.005,\,9} = 23.59$, $\chi^2_{0.995,\,9} = 1.74$ $\chi^2_{0.005,\,10} = 18.307$, $\chi^2_{0.995,\,10} =$ 2.156, choose appropriately.

(ii) The mean life time of a sample of 100 electric bulbs produced by a manufacturing company is estimated to be 1570 hours with a standard deviation of 120 hours. If μ be the mean life time of all the bulbs produced by the company, test the hypotheses $H : \mu = 1600$ hours against the alternative hypotheses $K : \mu \ne 1600$ hours, using a level of significance. It is given that $z_{0.025} = 1.96$, $z_{0.05} = 1.645$, choose appropriately. 5+5

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P.T.O.

OR

[Boolean Algebra and Automata Theory]

Group - A

1. Answer any ***ten*** questions : 2×10=20

(a) Define ordered sets and provide an example.

(b) What is the duality principle in ordered sets and how is it applied?

(c) Explain the concept of lattices as algebraic structures.

(d) Define sublattices and provide an example.

(e) Write the formal definition of Pushdown Automata.

(f) Describe the Quinn-McCluskey method for simplifying Boolean polynomials.

(g) Explain the concept of minimal and maximal forms of Boolean polynomials.

(h) What are Karnaugh diagrams and how are they used in Boolean algebra?

(i) Define logic gates and their role in switching circuits.

(j) Provide an application of switching circuits.

(k) Define alphabets, strings, and languages in the context of automata theory.

(l) What are regular expressions and their relationship with finite automata?

(m) Explain the pumping lemma for regular languages.

(n) Show that if P is a pushdown automata, then there is a one-state pushdown automata P', such that $N(P') = N(P)$.

(o) What is the significance of the halting problem in the context of Turing machines?

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Let P and Q be chains. Prove that $P \times Q$ is a chain in the lexicographic order. Prove that $P \times Q$ is a chain in the coordinate-wise order if and only if at most one of P and Q has more than one element.

(b) Prove that, for all ordered sets P, Q and R, $< P \to < Q \to R \gg \cong < P \times Q \to R >$.

(c) Discuss the role of context-free grammars in language generation and parsing.

P.T.O.

(d) Explain the concept of Turing machines as a model of computation and their variants.

(e) Discuss the significance of undecidability problems about Turing machines.

(f) Let L be a lattice. Prove that the following are equivalent :

(i) L is a chain;

(ii) every non-empty subset of L is a sublattice;

(iii) every two-element subset of L is a sublattice.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) Compare and contrast deterministic and non-deterministic finite automata, highlighting their differences in language acceptance capabilities.

(b) (i) Draw switching circuits for these Boolean expressions :

(A) $(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

(B) $\left[a \wedge ((b \wedge \sim c) \vee (\sim b \wedge c)\right) \vee (\sim a \wedge b \wedge c)$.

(ii) Establish a truth table for the Boolean function $f(x_1, x_2, x_3) = (\sim x_1 \vee x_2) \wedge (\sim x_3 \vee x_2)$. Draw a circuit using as few AND, OR, NOT gates as possible to model the function.

(c) Explain the concept of recursively enumerable and recursive languages, and their significance in the context of undecidability.

(d) Discuss the implications of undecidability problems about context-free grammars and their impact on language recognition.

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P.T.O.

OR

Group - A

[Portfolio Optimization]

1. Answer any ***ten*** questions from the following : 2×10=20

(a) What is primary market?

(b) Write difference between Security Market Line (SML) and Capital Market Line (CML).

(c) What is portfolio risk and return?

(d) What is Net Amount Value (NAV)?

(e) Which is better — Real Estate or Equity?

(f) Explain the term monetary policy.

(g) What is Portfolio Management Process?

(h) Write down the functions of SEBI.

(i) What is Annuity?

(j) Define diversification.

(k) Write the different types of risks.

(l) You save Rs. 200 and invest it at a nominal interest rate of 8%. Given the expected inflation is 6% per year. What is the real rate of return?

(m) What are the advantages of mutual fund?

(n) What do you mean by Portfolio Performance?

(o) Is a mutual fund with a low NAV better?

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Explain the structure of SEBI.

(b) Use the information in the following to answer the questions below.

State of Economy	Probability of state	Return on A in state	Return on B in state
Boom	35%	0.040	0.210
Normal	50%	0.030	0.080
Recession	15%	0.046	–0.010

What is the expected return of each asset?

(c) Define :

(i) Beta of a Portfolio

(ii) Capital market line

(d) You have a portfolio with a beta of 0.84. What will be the new portfolio beta if you keep 85% of your money in the old portfolio and 14% in a stock with a beta of 1.93?

(e) Explain with example — Portfolios with short sales.

(f) Write some of the benefits of diversification.

P.T.O.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) For the Markowitz mean-variance Portfolio solve the Quadratic Programming Problem.

Minimize $\frac{1}{2}W^T\Sigma W - \lambda m^T W$

Subject to $e^T W = 1$

where $W = (W_1, W_2, ..., W_n)^T$,

$m = (m_1, m_2, ..., m_n)^T$

$\mu_i = E(r_i)$, $z = (r_1, r_2, ..., r_n)^T$

$\text{cov}(z) = \Sigma$. 10

(b) Prove that the expected return μ_i on any asset i satisfies $\mu_i = r_f + \beta_i(\mu_M - r_f)$, where $\beta_i = \frac{\sigma_{iM}}{\sigma_M{}^2}$ and σ_{iM} is the covariance of the return on asset i and the market Portfolio r_M; $\sigma_M{}^2 = \text{var}(r_M)$. 10

(c) Consider 3 assets with rate of return r_1, r_2 and r_3 respectively the covariance matrix and expected rates of return are $\Sigma = (2\ 1\ 0\ 1\ 2\ 1\ 0\ 1\ 2)$ and $m = (0.4\ \ 0.4\ \ 0.8)$.

(i) Find the minimum variance portfolio.

(ii) Find a second efficient portfolio. 5+5

(d) Assume that the expected rate of return on the efficient market portfolio is 24% ($r_M = 0.24$) and the rate of return on T-Bills is 70% ($r_f = 0.07$). The standard deviation is 33% $(\sigma_M = 0.33)$. What is the equation for the capital market line? 10

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