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B.Sc./4th Sem (H)/MATH/23(CBCS)

2023

4th Semester Examination
MATHEMATICS (Honours)

Paper : C 8-T

[Riemann Integration and Series of Functions]

[CBCS]

Full Marks : 60

Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answer wherever necessary.
Symbols have their usual meaning.

1. Answer any ***ten*** questions of the following : $2\times10=20$

(i) Define a power series with real coefficients. Give an example of a nowhere convergent power series.

(ii) Find the radius of convergence of the power series

$$x+\frac{(2!)^2}{5!}x^2+\frac{(3!)^2}{7!}x^3+\cdots\frac{(n!)^2}{(2n+1)!}x^n+\cdots.$$

(iii) Given $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, $x\in(-\infty,\infty)$,

P.T.O.

deduce that there exists a unique number $\alpha, 0 < \alpha < 2$ such that $\cos\alpha = 0$.

(iv) If $f:[a, b] \to \mathbb{R}$ is bounded and increasing then prove that its lower and upper integrals are equal.

(v) A function f is defined on $[-2, 1]$ by $f(x) = \operatorname{sgn}(x)$ and $\phi(x) = |x|$. Show that $\int_{-2}^{1} f(x)dx = \phi(1) - \phi(-2)$.

(vi) Give an example of two integrable functions f and g on $[a, b]$ such that $f(x) > g(x)$ for all $x \in [a, b]$ but $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

(vii) Find whether $f(x) = \left|x - \frac{1}{2}\right|$ is R-integrable on $[0, 1]$.

(viii) Defining e by the relation $\int_1^e \frac{dt}{t} = 1$, show that $2 < e < 3$.

(ix) Explain when the improper integral $\int_a^\infty f(x)dx$ is said to be absolutely convergent. Is an absolutely convergent integral convergent?

(x) Prove that $\Gamma(n+1) = n\Gamma(n)$ for $n > 0$.

(xi) Explain when the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, $-\pi \leq x \leq \pi$ is said to be a Fourier series corresponding to a function $f(x)$ in $[-\pi, \pi]$. State the conditions under which the above series converges to $f(x)$.

(xii) Give an example to show that if $\{f_n(x)\}$ converges uniformly to $f(x)$ on $S(\subset \mathbb{R})$ and $\{g_n(x)\}$ converges uniformly to $g(x)$ on S then $\{f_n g_n(x)\}$ may not be uniformly convergent on S.

(xiii) Show that the sequence $\left\{|x|^{1+\frac{1}{n}}\right\}$, $-1 \leq x \leq 1$ is uniformly convergent on $[-1, 1]$.

(xiv) Show that $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-a, a]$ where $0 < a < 1$.

(xv) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \rho$. If $f(-x) = f(x)$ for all $x \in (-\rho, \rho)$ then show that $a_n = 0$ for all odd values of n.

2. Answer any ***four*** questions of the following : 5×4=20

(i) Show that $f(x) = x\left[x^2\right]$ is integrable on $[0, 2]$. Find $\int_0^2 f(x)dx$.

P.T.O.

(ii) Show that the series $\sum_{k=0}^{\infty}(-1)^k x^{2k}$ may be integrated term by term from 0 to h, $-1<h<1$ and thus prove the validity of the Maclaurin's expansion of $\tan^{-1} h$ for $-1<h<1$. Hence or otherwise deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

(iii) Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$ is convergent when $m>0$, $n>-1$.

(iv) If $f(x)$ be continuous in $[a, b]$ and $g(x)$ be integrable in $[a, b]$ retaining the same sign throughout $[a, b]$, show that there exists a point $p\in[a, b]$ such that

$$\int_a^b f(x)g(x)\,dx=f(p)\int_a^b g(x)\,dx.$$

Give suitable example to show that the condition of continuity cannot be dropped.

(v) State second MVT of Integral Calculus in (a) Bonnet's form (b) Weirstrass's form. Discuss their applicability for $f(x)=\cos x$, $g(x)=x^2$; $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

(vi) If $f(x)=\begin{cases}\pi/3 & 0\le x\le \pi/3\\ 0 & \pi/3<x<2\pi/3\\ -\pi/12 & x=2\pi/3\\ -\pi/3 & 2\pi/3<x\le\pi\end{cases}$

Find the Fourier cosine series of f on $[0, \pi]$.

Hence deduce that $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\cdots=\frac{\pi}{2\sqrt{3}}$.

3. Answer any *two* questions of the following : 10×2=20

(i) (a) Show that $\frac{\pi^3}{192}<\int_0^{\pi/2}\frac{x^2}{5+3\cos x}\,dx<\frac{\pi^3}{120}$.

(b) A function f is defined on [0, 1] by

$$f(x)=x^3,\quad x\in[0,1]\cap Q$$
$$=2x^2\quad x\in[0,1]-Q.$$

Calculate two different Riemann Sums. Show that f is not integrable on [0, 1]. 5+5=10

(ii) (a) Let $f,\ g:[a, b]\to\mathbb{R}$ be both bounded over $[a, b]$ such that $f(x)=g(x)$ except for a finite number of points in $[a, b]$. If f is R-integrable over $[a, b]$, prove that g is also R-integrable over $[a, b]$ and $\int_a^b f(x)\,dx=\int_a^b g(x)\,dx$.

(b) Show that if $x>0$, $\frac{x}{1+x}<\log(1+x)<x$.

7+3=10

P.T.O.

(iii) (a) If $f(x)=(\pi-|x|)^2$ on $[-\pi, \pi]$, prove that the Fourier series of f is given by

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx.$$

Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

(b) With proper justification, show that $\lim\limits_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$.

6+4=10

(iv) (a) Test the convergence of $\int_1^{\infty}\frac{\cos ax-\cos bx}{x}dx$.

(b) Show that $\int_2^{\infty}\frac{x^2}{\sqrt{(x^7+1)}}dx$ is convergent but $\int_2^{\infty}\frac{x^3}{\sqrt{(x^7+1)}}dx$ is divergent.

(c) Show that $\int_0^{\infty}\frac{\sin x}{x}dx$ converges conditionally.

3+4+3=10
