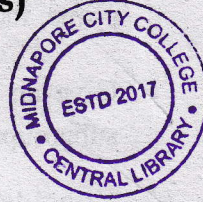


2023

3rd Semester Examination
MATHEMATICS (Honours)

Paper : C 6-T
(Group Theory - 1)
[CBCS]



Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

Answer any *ten* questions : $2 \times 10 = 20$

1. Let S be any set and $P(S)$ be the power set of S . Define an operation on $P(S)$ to make it an abelian group. What is identity element in this group and how is the inverse of any other element in this group?
2. Find the inverse of $\bar{13}$ in the group $(\mathbb{Z}_{20}, +)$.
3. Suppose that $(ab)^n = a^n b^n$ for all $a, b \in G$, $n > 1$ is a fixed integer. Prove that $(ab)^{n-1} = b^{n-1} a^{n-1}$.

P.T.O.

(2)

4. Find the order of the element $\langle 6 \rangle + 5$ in the group $\frac{\mathbb{Z}_8}{\langle 6 \rangle}$.
5. If H and K are two subgroups whose order are relatively prime, then show that $H \cap K = \{e\}$.
6. Give an example to prove that union of two proper subgroups need not be a subgroup always.
7. Considering the group S_{10} . Find the order of $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 4 & 2 & 6 & 9 & 10 & 3 & 5 & 8 \end{pmatrix}$.
8. Let G be a group of order 11. Find all non-isomorphic subgroups of G .
9. Prove that any infinite cyclic group is isomorphic to the additive group \mathbb{Z} of all integers.
10. Let G be a group of order 35 and A and B be subgroups of G of order 5 and 7 respectively. Show that $G = AB$.
11. Let $G = \langle a \rangle$ be a cyclic group of order 30. Then prove that $[G : \langle a^5 \rangle] = 5$.
12. Consider the groups $Z_2 = \{0, 1\}$ and $Z_3 = \{0, 1, 2\}$ under addition modulo. Then find $Z_2 \times Z_3$. If $Z_2 \times Z_3$ is cyclic group, what is its generator?
13. Is union of two normal subgroups a normal subgroup again? Give logic in support of your answer.

(3)

14. Let G be a group generated by a, b such that $O(b) = 2$, $O(a) = 6$ and $(ab)^2 = e$. Show that $(a^2b)^2 = e$.
15. Let G be a group and H be its subgroup. Prove that $hH = Hh = H$ iff $h \in H$.

Group - B

Answer any four questions : 5×4=20

16. Let G be a group and $a \in G$ is of order n . Then prove that $O(a^p) = \frac{n}{\gcd(n, p)}$.
17. Consider S_4 . Let G be the subgroup of S_4 such that G is generated by the permutations $a = (1 2 3 4)$ and $b = (2 4)$. Show that G is a Dihedral group of degree 4.
18. Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \in \mathbb{Q}^* \right\}$, where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$. Then prove that G is an abelian group with respect to multiplication of matrices.
19. Let H and K be two finite subgroups of a group G . Prove that $O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$. Hence prove that if $O(H) > \sqrt{O(G)}$ and $O(K) > \sqrt{O(G)}$, then $H \cap K \neq \{e\}$.

P.T.O.

(4)

20. (i) Is $\mathbb{Z}_4 \times \mathbb{Z}_4$ a cyclic group? Give logic in support of your answer.

(ii) Does there exist a group epimorphism $\phi: \mathbb{Z}_4 \times \mathbb{Z}_4 \rightarrow \mathbb{Z}_{16}$? Give reason. 2+3

21. Let G be a group of order 8 and x be an element of G of order 4. Prove that $x^2 \in Z(G)$.

Group - C

Answer any two questions : 10×2=20

22. (i) State and prove Third theorem of Isomorphism.

(ii) Let (G, \circ) be a group and H be a non-empty finite subset of G . Then show that (H, \circ) is a subgroup of (G, \circ) if and only if $a \in H$, $b \in H \Rightarrow a \circ b \in H$. 10

23. (i) State and prove Lagrange's Theorem. Is the converse of Lagrange's theorem true? Justify with example.

(ii) If n , $a \in \mathbb{N}$ and $(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$ where $\phi(n)$ is the number of positive integers less than n and prime to n . 6+4

(5)

24. (i) Construct a group G of order 12 such that G has more than one subgroup of order 6.

(ii) State and prove Cayley's Theorem for groups. 4+(1+5)

25. (i) Let H_1 and H_2 be two normal subgroups of a group G . Then prove that G is an IDP of H_1 and H_2 if and only if (i) $G = H_1 H_2$,

(ii) $H_1 \cap H_2 = \{e\}$.

(ii) Consider the groups $\mathbb{Z}_2 \times \mathbb{S}_3$, $\mathbb{Z}_2 \times \mathbb{S}_6$ and \mathbb{Z}_{12} . Are any two of these groups isomorphic? Is any one noncommutative? 5+5