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2023

3rd Semester Examination

MATHEMATICS (Honours)

Paper : C 6-T

(Group Theory - 1)

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : $2\times10=20$

1. Let S be any set and $P(S)$ be the power set of S. Define an operation on $P(S)$ to make it an abelian group. What is identity element in this group and how is the inverse of any other element in this group?

2. Find the inverse of $\overline{13}$ in the group $(\mathbb{Z}_{20}, +)$.

3. Suppose that $(ab)^n = a^n b^n$ for all $a, b \in G$, $n > 1$ is a fixed integer. Prove that $(ab)^{n-1} = b^{n-1}a^{n-1}$.

P.T.O.

4. Find the order of the element $\langle 6\rangle+5$ in the group $\frac{Z_8}{\langle 6\rangle}$.

5. If H and K are two subgroups whose order are relatively prime, then show that $H\cap K=\{e\}$.

6. Give an example to prove that union of two proper subgroups need not be a subgroup always.

7. Considering the group S_{10}. Find the order of $p=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\7&1&4&2&6&9&10&3&5&8\end{pmatrix}$.

8. Let G be a group of order 11. Find all non-isomorphic subgroups of G.

9. Prove that any infinite cyclic group is isomorphic to the additive group Z of all integers.

10. Let G be a group of order 35 and A and B be subgroups of G of order 5 and 7 respectively. Show that $G=AB$.

11. Let $G=\langle a\rangle$ be a cyclic group of order 30. Then prove that $[G:\langle a^5\rangle]=5$.

12. Consider the groups $Z_2=\{0,1\}$ and $Z_3=\{0,1,2\}$ under addition modulo. Then find $Z_2\times Z_3$. If $Z_2\times Z_3$ is cyclic group, what is its generator?

13. Is union of two normal subgroups a normal subgroup again? Give logic in support of your answer.

14. Let G be a group generated by a, b such that $O(b)=2$, $O(a)=6$ and $(ab)^2=e$. Show that $(a^2b)^2=e$.

15. Let G be a group and H be its subgroup. Prove that $hH=Hh=H$ iff $h\in H$.

Group - B

Answer any ***four*** questions : 5×4=20

16. Let G be a group and $a\in G$ is of order n. Then prove that $O(a^p)=\frac{n}{\gcd(n,p)}$.

17. Consider S_4. Let G be the subgroup of S_4 such that G is generated by the permutations $a=(1\,2\,3\,4)$ and $b=(2\,4)$. Show that G is a Dihedral group of degree 4.

18. Let $G=\left\{\begin{pmatrix}a&a\\a&a\end{pmatrix}\middle| a\in Q^*\right\}$, where $Q^*=Q-\{0\}$. Then prove that G is an abelian group with respect to multiplication of matrices.

19. Let H and K be two finite subgroups of a group G. Prove that $O(HK)=\frac{O(H).O(K)}{O(H\cap K)}$. Hence prove that if $O(H)>\sqrt{O(G)}$ and $O(K)>\sqrt{O(G)}$, then $H\cap K\neq\{e\}$.

P.T.O.

20. (i) Is $\mathbb{Z}_4 \times \mathbb{Z}_4$ a cyclic group? Give logic in support of your answer.

(ii) Does there exist a group epimorphism $\phi: \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_{16}$? Give reason. 2+3

21. Let G be a group of order 8 and x be an element of G of order 4. Prove that $x^2 \in Z(G)$.

Group - C

Answer any ***two*** questions : 10×2=20

22. (i) State and prove Third theorem of Isomorphism.

(ii) Let $(G, \circ)$ be a group and H be a non-empty finite subset of G. Then show that $(H, \circ)$ is a subgroup of $(G, \circ)$ if and only if $a \in H$, $b \in H \Rightarrow a \circ b \in H$. 10

23. (i) State and prove **Lagrange's Theorem**. Is the converse of Lagrange's theorem true? Justify with example.

(ii) If n, $a \in N$ and $(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (\bmod n)$ where $\phi(n)$ is the number of positive integers less than n and prime to n. 6+4

24. (i) Construct a group G of order 12 such that G has more than one subgroup of order 6.

(ii) State and prove Cayley's Theorem for groups. 4+(1+5)

25. (i) Let H_1 and H_2 be two normal subgroups of a group G. Then prove that G is an IDP of H_1 and H_2 if and only if (i) $G = H_1 H_2$, (ii) $H_1 \cap H_2 = \{e\}$.

(ii) Consider the groups $Z_2 \times S_3$, $Z_2 \times S_6$ and Z_{12}. Are any two of these groups isomorphic? Is any one noncommutative? 5+5
