B.Sc./3rd Sem (H)/MATH/23(CBCS)

2023

3rd Semester Examination

MATHEMATICS (Honours)

Paper : C 5-T

(Theory of Real Functions and Introduction to Metric Space)

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any *ten* questions : 2×10=20

1. Evaluate : $\lim\limits_{x\to 3}\left([x]-\left[\dfrac{x}{3}\right]\right)$.

2. Give an example to show that a function which is continuous on an open bounded interval may not be uniformly continuous there.

3. Give the geometrical interpretation of MVT.

4. On the real line $\mathbb{R}$, show that a singleton set is not an open set.

P.T.O.

5. Using intermediate value theorem show that $x^3+x^2-x+1=0$ has a solution in the interval $(-2, 2)$.

6. Determine the points of discontinuity of $[\sin x]$ in $[-2\pi, 2\pi]$.

7. Let $f:[0, 1]\to\mathbb{R}$ be a continuous function that does not assume any of its values twice and with $f(0)<f(1)$. Show that f is strictly increasing on $[0, 1]$.

8. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$f(x)=x^2$, if x is rational

$=0$, if x is irrational.

Show that f is differentiable at $x=0$ and find $f'(0)$.

9. Examine if $f(x)=x-[x]$ has a local maximum or local minimum at $x=0$.

10. Use Taylor's theorem to prove that

$$1+\frac{x}{2}-\frac{x^2}{8}<\sqrt{1+x}<1+\frac{x}{2}, \text{ if } x>0.$$

11. $f: R\to R$ is defined by $f(x)=|x|+|x-1|+|x-2|, x\in R$. Find the derived function f' and specify the domain of f'.

12. Prove that in any discrete metric space any set is an open set.

13. Let (M, d) be a metric space. Then for all $A, B\subset M$, prove that $A\cap B\neq\varphi\Rightarrow\delta(A\cup B)\leq\delta(A)+\delta(B)$, where $\delta(A)$ represents diameter of A.

14. In the set R^2 of all order pair of real numbers consider the function $d(X,Y)=|x_1-y_1|+|x_2-y_2|$ for all $X=(x_1, x_2)$, $Y=(y_1, y_2)\in R^2$. Prove that (R^2, d) is a metric space.

15. Let (X, d) be a metric space. Then prove that $\forall A$, $B\subseteq X$, $A\subset B\Rightarrow S(A)\leq S(B)$, where $S(A)$ denotes the diameter of A.

Group - B

Answer any ***four*** questions : 5×4=20

16. Prove that $\frac{x}{\sin x}<\frac{\tan x}{x}$ for $x\in(0, \pi/2)$.

17. State and prove the Hausdorff Property.

18. Let I be a closed and bounded interval and let $f: I\to\mathbb{R}$ be continuous on I. Then prove that f is uniformly continuous on I.

19. Let $I\subseteq\mathbb{R}$ be an interval and let $f: I\to\mathbb{R}$ be strictly monotone and continuous on I. Prove that the function g inverse to f is strictly monotone and continuous on $f(I)$.

P.T.O.

20. State and prove Darboux's theorem on differentiability. 1+4

21. Define metric space. Let A be a non-empty set and d_1, d_2 be two metrics defined on it. Prove that a function $d: A \times A \rightarrow R$ given by $d = \max\{d_1, d_2\}$ is another metric on A.

Group - C

Answer any ***two*** questions : 10×2=20

22. (i) Let (A, d) be a metric space. Then prove that $(A, \sqrt{d})$ is also a metric space.

(ii) In any metric space (X, d), prove that intersection of a finite number of open sets is open. 4+6

23. (i) State and prove Lagrange's mean value theorem.

(ii) Find the Taylor series for $f(x)$ centered at the given value of a, where $f(x) = 2\cos x$, $a = 7\pi$. 6+4

24. (i) A function $f: [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = x, \text{ if } x \text{ is rational in } [0, 1]$$
$$= (1 - x), \text{ if } x \text{ is irrational in } [0, 1].$$

Prove that

(a) f is injective on [0, 1],

(b) f assumes every real number in [0, 1],

(c) f is continuous only at $x = \frac{1}{2}$ in [0, 1] and discontinuous at every other point in [0, 1].

(ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) = 0\}$ is closed in $\mathbb{R}$. (2+2+3)+3

25. (i) Let $D \subset R$ and $f: D \rightarrow R$ be a function. Let $c \in D'$. Then prove that a necessary and sufficient condition for the existence of $\lim_{x \to c} f(x)$ is that for a pre-assigned positive ε there exist a positive δ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair of points $x_1, x_2 \in N'(c, \delta) \cap D$.

(ii) Let the functions $f: R \rightarrow R$ and $g: R \rightarrow R$ are both continuous on R. Then prove that the set $S = \{x \in R : f(x) = g(x)\}$ is a closed set in R. 6+4
