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B.Sc./3rd Sem (H)/MATH/23(CBCS)

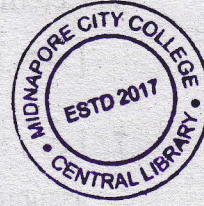
2023

3rd Semester Examination
MATHEMATICS (Honours)

Paper : C 5-T

(Theory of Real Functions and
Introduction to Metric Space)

[CBCS]



Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

Answer any *ten* questions : $2 \times 10 = 20$

1. Evaluate : $\lim_{x \rightarrow 3} \left([x] - \left[\frac{x}{3} \right] \right)$.
2. Give an example to show that a function which is continuous on an open bounded interval may not be uniformly continuous there.
3. Give the geometrical interpretation of MVT.
4. On the real line \mathbb{R} , show that a singleton set is not an open set.

P.T.O.



(2)

5. Using intermediate value theorem show that $x^3 + x^2 - x + 1 = 0$ has a solution in the interval $(-2, 2)$.

6. Determine the points of discontinuity of $[\sin x]$ in $[-2\pi, 2\pi]$.

7. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function that does not assume any of its values twice and with $f(0) < f(1)$. Show that f is strictly increasing on $[0, 1]$.

8. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by
 $f(x) = x^2$, if x is rational
 $= 0$, if x is irrational.

Show that f is differentiable at $x = 0$ and find $f'(0)$.

9. Examine if $f(x) = x - [x]$ has a local maximum or local minimum at $x = 0$.

10. Use Taylor's theorem to prove that
 $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

11. $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by
 $f(x) = |x| + |x-1| + |x-2|$, $x \in \mathbb{R}$. Find the derived function f' and specify the domain of f' .

12. Prove that in any discrete metric space any set is an open set.



(3)

13. Let (M, d) be a metric space. Then for all $A, B \subset M$, prove that $A \cap B \neq \emptyset \Rightarrow \delta(A \cup B) \leq \delta(A) + \delta(B)$, where $\delta(A)$ represents diameter of A .

14. In the set \mathbb{R}^2 of all order pair of real numbers consider the function $d(X, Y) = |x_1 - y_1| + |x_2 - y_2|$ for all $X = (x_1, x_2)$, $Y = (y_1, y_2) \in \mathbb{R}^2$. Prove that (\mathbb{R}^2, d) is a metric space.

15. Let (X, d) be a metric space. Then prove that $\forall A, B \subseteq X$, $A \subset B \Rightarrow S(A) \subseteq S(B)$, where $S(A)$ denotes the diameter of A .

Group - B

Answer any four questions :

5×4=20

16. Prove that $\frac{x}{\sin x} < \frac{\tan x}{x}$ for $x \in (0, \pi/2)$.

17. State and prove the Hausdorff Property.

18. Let I be a closed and bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Then prove that f is uniformly continuous on I .

19. Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I . Prove that the function g inverse to f is strictly monotone and continuous on $f(I)$.

P.T.O.

(4)

20. State and prove Darboux's theorem on differentiability.

1+4

21. Define metric space. Let A be a non-empty set and d_1, d_2 be two metrics defined on it. Prove that a function $d: A \times A \rightarrow \mathbb{R}$ given by $d = \max\{d_1, d_2\}$ is another metric on A .

Group - C

Answer any two questions : 10×2=20

22. (i) Let (A, d) be a metric space. Then prove that (A, \sqrt{d}) is also a metric space.

(ii) In any metric space (X, d) , prove that intersection of a finite number of open sets is open. 4+6

23. (i) State and prove Lagrange's mean value theorem.

(ii) Find the Taylor series for $f(x)$ centered at the given value of a , where $f(x) = 2 \cos x, a = 7\pi$.

6+4

24. (i) A function $f: [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = x, \text{ if } x \text{ is rational in } [0, 1]$$

$$= (1-x), \text{ if } x \text{ is irrational in } [0, 1].$$

Prove that

(a) f is injective on $[0, 1]$,

(5)

(b) f assumes every real number in $[0, 1]$,

(c) f is continuous only at $x = \frac{1}{2}$ in $[0, 1]$ and discontinuous at every other point in $[0, 1]$.

(ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on \mathbb{R} . Prove that the set $S = \{x \in \mathbb{R}: f(x) = 0\}$ is closed in \mathbb{R} . (2+2+3)+3

25. (i) Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$ be a function. Let $c \in D$. Then prove that a necessary and sufficient condition for the existence of $\lim_{x \rightarrow c} f(x)$ is that for a pre-assigned positive ε there exist a positive δ such that $|f(x_1) - f(x_2)| < \varepsilon$ for every pair of points $x_1, x_2 \in N'(c, \delta) \cap D$.

(ii) Let the functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are both continuous on \mathbb{R} . Then prove that the set $S = \{x \in \mathbb{R}: f(x) = g(x)\}$ is a closed set in \mathbb{R} .

6+4

