বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2023

(Under CBCS Pattern)

Semester — II

Subject : MATHEMATICS

Paper : C-4T

(Differential Equations and Vector Calculus)

Full Marks : 60

Time : 3 hours

The figures in the right-hand margin indicate marks.

The symbols used have their usual meanings.

Answer from **all** the Groups as directed.

GROUP—A

1. Answer *any* **ten** questions from the following : 2×10=20

   (a) Are these three functions e^x, e^{2x}, e^{3x} linearly independent? Justify.

(b) State the existence and uniqueness theorem for the nth order linear homogeneous differential equation.

(c) Find the particular integral of

$$(D^2+4)y = e^x \sin 2x, \text{ where } D \equiv \frac{d}{dx}.$$

(d) Define isolated critical point of the linear autonomous system of differential equations.

(e) Find a necessary condition for the vector $\vec{c}(t)$ to be constant.

(f) If e^{2x} and xe^{2x} are particular solutions of a second order homogeneous differential equation with constant coefficients, then find the equation.

(g) Scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is the volume of a parallelepiped. Justify.

(h) Distinguish between initial value problem and boundary value problem.

(i) Prove that

$$[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

(j) Find the first order simultaneous differential equations for the third order differential equation of

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 18e^{2x}$$

(k) Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at (3, 1, 2).

(l) Consider a system of differential equations $\frac{dx}{dt} = 2x - 7y$, $\frac{dy}{dt} = 3x - 8y$. Discuss the nature of critical point at (0, 0) of the system.

(m) Find a unit normal to the surface

$$x^2 - y^2 + z = 2, \text{ at } (1, -1, 2)$$

(n) Test the continuity of the vector function $\vec{r} = |t|\,\hat{i} - \sin t\,\hat{j} + (1+\cos t)\hat{k}$ at $t = 0$.

(o) If $\vec{A} = x^2 y\,\hat{i} - xz\,\hat{j} + 4yz\hat{k}$, then find div(curl $\vec{A}$).

GROUP—B

2. Answer *any* **four** questions from the following : 5×4=20

(a) Solve $\frac{dx}{dt} = x^2 + xy$, $\frac{dy}{dt} = y^2 + xy$ satisfying the initial conditions $x = 1, y = 2$ when $t = 0$.

(b) Prove that

$$\left(\vec{\alpha} \times \vec{\beta}\right) \cdot \left(\vec{\gamma} \times \vec{\delta}\right) + \left(\vec{\alpha} \times \vec{\gamma}\right) \cdot \left(\vec{\delta} \times \vec{\beta}\right) + \left(\vec{\alpha} \times \vec{\delta}\right) \cdot \left(\vec{\beta} \times \vec{\gamma}\right) = 0$$

(c) Find the maximum value of the directional derivative at $(1, 1, -1)$ of $\phi = x^2 - 2y^2 + 4z^2$ in the direction $(2\hat{i} + \hat{j} - \hat{k})$.

(d) Solve

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$$

(e) Suppose $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$,

$$\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$$

show that $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -4\hat{i} - 8\hat{j}$ at $(1,0,-2)$.

(f) Solve the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x, \; x > 0$$

by the method of variation of parameters.

GROUP—C

3. Answer *any* **two** questions from the following : 10×2=20

(a) Consider the autonomous system

$$\frac{dx}{dt} = x + y + x^2y$$

$$\frac{dy}{dt} = 3x - y + 2xy^3$$

(i) Determine the type of the critical point $(0, 0)$.

(ii) Obtain the differential equation of the paths and find its general solution.

(iii) Carefully plot the paths to obtain a phase plane diagram which clearly exhibits the nature of the critical point $(0, 0)$. 10

(b) (i) Show that the vector

$$\vec{V} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$$

is irrotational. Show that $\vec{V}$ can be expressed as the gradient of some scalar function ϕ.

(ii) If $\vec{r}(t) = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$, then prove that

$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2} \right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

5+5=10

(c) What do you mean by singular point? Use the method of Frobenius to find the solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4} \right) y = 0$$ 2+8=10

(d) *(i)* Solve $\dfrac{dy}{dt} + \dfrac{dx}{dt} - x - 6y = e^{3t}$ and

$$2\frac{dy}{dt} + \frac{dx}{dt} - 2x - 6y = t$$

(ii) Find the work done by a moving particle in the field of force

$$\vec{F} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\,\hat{k},$$

along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$. 6+4=10

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