বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2023

(Under CBCS Pattern)

Semester — II

Subject : MATHEMATICS

Paper : C–3T

(Real Analysis)

Full Marks : 60

Time : 3 hours

The figures in the right-hand margin indicate marks.
The symbols used have their usual meanings.

Answer **all** questions.

1. Answer *any* **ten** questions from the following : $2\times10=20$

(a) Define enumerable set. Is the set $Z=\{0,\pm1,\pm2,\pm3,.........\}$ of all integers enumerable (countable)? Justify.

(b) Prove that every convergent sequence is bounded.

(c) Is the sequence $\left\{\frac{3n+1}{n+2}\right\}$ bounded?

(d) Prove that the set Q of all rational numbers is neither open nor closed.

(e) What is meant by the following?

$$\lim_{n\to\infty} x_n = \infty$$

where $\{x_n\}$ is a sequence of real numbers.

(f) Show that $\sqrt{17}$ is not a rational number.

(g) Use definition of limit of a sequence to show that the sequence $\left\{\frac{1}{n}\right\}$ converges to zero.

(h) Find the derived set of

$$\left\{\frac{2}{p}+\frac{3}{q} \mid p, q = 1, 2, 3, \ldots\right\}$$

(i) Give an example to show that union of an arbitrary family of closed sets is not a closed set.

(j) Determine GLB and LUB of the following set :

$$\left\{\frac{3}{2}, \frac{5}{2}, \frac{5}{3}, \frac{7}{3}, \frac{7}{4}, \frac{9}{4}, \frac{9}{5}, \frac{11}{5}, \ldots\right\}$$

(k) Show that the series

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$$

is convergent and find its sum.

(l) State Cauchy's principle for the convergence of an infinite series.

(m) What do you mean by conditionally convergent of a series? Give example.

(n) Examine the convergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

(o) Use root test to examine the convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots$$

2. Answer *any* **four** questions from the following : 5×4=20

(a) If $\{a_n\}$ is a monotonic increasing sequence of real numbers, then prove that $\lim_{n\to\infty} a_n = \sup\{a_n\}$.

(b) Use Cauchy's principle of convergence to prove that $\{u_n\}$ is convergent, where

$$u_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

(c) Show that if $x, y \in R$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

(d) Examine whether the series is convergent or divergent :

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \cdots$$

(e) Prove that the set $(0,1)$, is not enumerable. Hence show that the set R of real numbers is not enumerable.

(f) Investigate the convergence or divergence of any one of the following series

(i) $\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$

3. Answer *any* **two** questions from the following : 10×2=20

(a) (i) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is monotone increasing and bounded above.

(ii) If a sequence $\{x_n\}$ of real numbers is monotone increasing but not bounded above, then prove that it diverges to $+\infty$.

6+4=10

(b) (i) If Σu_n and Σv_n be two series with non-negative terms and suppose that there exists an integer N, such that $u_n \le v_n, \forall n > N$, then prove that

(1) Σu_n converges if Σv_n converges,

(2) Σv_n diverges if Σu_n diverges

(ii) Examine the convergence of

$$2 + \frac{3}{8} + \frac{4}{27} + \frac{5}{64} + \cdots + \frac{(n+1)}{n^3} + \cdots$$

(3+3)+4=10

(c) (i) Define open set. Prove that the union of an arbitrary collection of open sets is an open set. Is the intersection of an arbitrary collection of open sets open? Justify your answer.

(ii) State Heine-Borel theorem. Give an illustration of it. (1+3+1+2)+1+2=10

(d) (i) If a set S is open, then its complement S^c is closed and conversely, justify this statement.

(ii) State Bolzano-Weierstrass theorem. Define 'covering' of a set. Let $S = (0,1)$ be a set, find an open covering of S.

5+5=10

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