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B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : C 13-T

[Metric Spaces and Complex Analysis]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** questions : $2\times10=20$

1. State the Banach fixed point theorem.
2. What do you mean by complete metric space?
3. Define uniform continuity.
4. Write down the Heine-Borel property.
5. Let (X, d) be a metric space in which A and B are two intersecting connected sets. Show that $A \cup B$ is connected.

P.T.O.

6. Let $f(z) = \frac{|z|}{\operatorname{Re}(z)}$ if $\operatorname{Re}(z) \neq 0$

$= 0$ if $\operatorname{Re}(z) = 0$

Show that $f(z)$ is not continuous at $z = 0$.

7. Show that the function $u = \cos x \cosh y$ is harmonic.

8. Find the radius of convergence $\sum_{h=2}^{\infty} \frac{z^n}{n.(\log n)^2}$.

9. Evaluate $\oint_c \frac{1}{z} dz$, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

10. Show that $u(x, y) = 4xy - x^3 + 3xy^2$, is harmonic.

11. Show that a convergent sequence in a metric space is bounded.

12. Give an example, in the real time $\mathbb{R}$, of the sequence $\{x_n\}$ such that $|x_n - x_{n+1}| \to 0$ (as $n \to \infty$) but $\{x_n\}$ is not Cauchy.

13. Show that for any subset A of a metric space (X, d), the function $f: X \to \mathbb{R}$ given by $f(x) = d(x, A)$, $x \in X$, is uniformly continuous.

14. Show that $\lim_{z \to z_0} f(z) g(z) = 0$ if $\lim_{z \to z_0} f(z) = 0$ and if there exists a positive integer M such that $|g(z)| \le M$ for all z in some neighbourhood of z_0.

15. Let $T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation. Show that ∞ is a fixed point of T if and only if $c = 0$.

Group - B

Answer any ***four*** questions : 5×4=20

16. Show that continuous image of a compact metric space is compact.

17. Check whether the function is differentiable at $z = 0$. Also check whether it satisfies C-R equations.

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} \quad (z \neq 0)$$
$$= 0 \quad (z = 0)$$

18. If $f(z)$ is an alytic function within and on a closed contour C, and if a is any point within C, then show that

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

19. (i) Prove that a metric space (X, d) having the property that every continuous map $f: X \to X$ has a fixed point, is connected. 2

(ii) Let (X, d) be a complete metric space and $T: X \to X$ be a contraction on X. Then for $x \in X$, show that the sequence $\{T^n(x)\}$ is a convergent sequence. 3

P.T.O.

20. (i) Determine whether the set $S=\{(x,y):0<x\le 1, x^2+y^2=4\}$ is compact in $\mathbb{R}^2$. 3

(ii) Let X be an infinite set endowed with the discrete metric. Show that every infinite subset of (X, d) is bounded but not totally bounded. 2

21. Evaluate :

(i) $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$ where C is the circle $|z|=3$ described in the positive sence. 2

(ii) $\int_C \frac{zdz}{(9-z^2)(z+i)}$ where C is the circle $|z|=2$ described in the positive sence. 3

Group - C

Answer any ***two*** questions : 10×2=20

22. (i) State and prove Liouville's theorem.

(ii) If $f(z)$ is a regular analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$.

23. (i) Show that any compact subset of a metric space is closed and bounded.

(ii) Show that two metrics d_1, d_2 on a set X are equivalent iff the identity map $I_X:(X,d_1)\to(X,d_2)$ is a homomorphism.

24. (i) Show that the map $f:[0,1]\to[0,1]$, defined by $f(x)=x-\frac{x^2}{2}$, $x\in[0,1]$ is a weak contraction but not a contraction map. 3

(ii) Let (X, d) be a complete metric space and $f:X\to X$ be a contraction map with Lipschitz constant t $(0<t<1)$. If $x_0\in X$ is the unique fixed point of f, show that $d(x,x_0)\le\frac{1}{1-t}d(x,f(x))$, for all $x\in X$. 5

(iii) Show that a contraction of a bounded plane set may have the same diameter as the set itself. 2

25. (i) Let $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$ and $z_0=x_0+iy_0$. Let the function f be defined in a domain D except possibly at the point z_0 in D. Then prove that $\lim_{z\to z_0} f(z)=u_0+iv_0$ if and only if $\lim_{x\to x_0}u(x,y)=u_0$ and $\lim_{y\to y_0}v(x,y)=v_0$. 5

(ii) Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2}=\frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}.$$ 5
