B.Sc./5th Sem (H)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (Honours)

Paper : C 12-T

[Group Theory II]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any ***ten*** questions : $2\times10=20$

(a) Prove that the automorphism group of $(\mathbb{Z}_8, +)$ is the Klein's Four Group.

(b) Prove that the mapping $f:U_{16}\to U_{16}$ sending $x\mapsto x^3$ is an automorphism.

(c) Let G_1 and G_2 be two groups. Then show that $Z(G_1\times G_2)=Z(G_1)\times Z(G_2)$.

P.T.O.

(d) Show that $n\mathbb{Z}$ is a characteristic subgroup of $(\mathbb{Z}, +)$ for any positive integer n.

(e) Compute the commutator subgroup of S_3.

(f) Is the direct product $\mathbb{Z} \times \mathbb{Z}$ a cyclic group? Justify your answer.

(g) For any integer $n > 1$, give examples of two non-isomorphic groups of order n^2.

(h) Is the group $\mathbb{Z}_{75} \oplus \mathbb{Z}_{20}$ isomorphic to $\mathbb{Z}_{25} \oplus \mathbb{Z}_{60}$? Justify your answer.

(i) Consider the left action of the group $G = GL(2, \mathbb{R})$ on $G = GL(2, \mathbb{R})$ by conjugation.

Find the orbit of $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

(j) Give an example (with proper justification) of a group action on a group G so that the kernel of the action is the centre of G.

(k) Let H be a subgroup of order 13 and index 5 of a group G. Prove that H is normal in G.

(l) State Sylow's first theorem.

(m) Using Sylow's Theorem, prove that a group of order 88 has a normal subgroup of order 11.

(n) Does there exist a group whose class equation is $10 = 1 + 1 + 1 + 2 + 5$? Give reason in support of your answer.

(o) Prove that any p-group has non-trivial centre.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Prove that the automorphism group of $(\mathbb{Q}, +)$ is isomorphic with $(\mathbb{Q}^*, \cdot)$.

(b) Find out all the conjugate classes of S_4.

(c) Let G be a group of order $2m$, where m is an odd integer. Show that G has a normal subgroup of order m.

(d) Show that the mapping $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f((a, b)) = a - b$ is a group homomorphism. Find the kernel of f. 3+2

(e) Let G be a group of order pq, where p, q are primes such that $p > q$ and q does not divide $p - 1$. Then show that G is a cyclic group.

(f) Prove that a group of order 96 has a normal subgroup of order 16 or 32.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) Compute the automorphism group of S_3.

(ii) Justify whether the following statement is true or not — "$\mathbb{Z}_4 \times \mathbb{Z}_6$ is isomorphic with $\mathbb{Z}_{24}$".

(iii) Find the number of elements of order 7 in a group of order 14. 4+2+4

(b) (i) Let H and K be two subgroups of a group G such that $H \subseteq K$. Then prove that

(1) if H is a characteristic subgroup of K and K is a characteristic subgroup of G then H is a characteristic subgroup of G, and

(2) if H is a characteristic subgroup of K and K is a normal subgroup of G then H is a normal subgroup of G.

(ii) Consider the action of the permutation group $G = \{(1), (1\ 2\ 3), (1\ 3\ 2), (4\ 5), (1\ 2\ 3)(4\ 5), (1\ 3\ 2)(4\ 5)\}$ on the set $I_5 = \{1, 2, 3, 4, 5\}$ defined by $(\sigma, a) \mapsto \sigma(a)$ for all $\sigma \in G$ and for all $a \in I_5$. Then find the stabilizers of 1 and 4.

(iii) Is conjugate of a Sylow p-subgroup of a group G again a Sylow p-subgroup of G? Justify your answer. (3+3)+2+2

(c) (i) Let G be a group acting on a set S. Then show that this group action induces a homomorphism from G to $A(S)$ where $A(S)$ denotes the permutation group of S. Hence prove that if H is a subgroup of a group G and S be the set of all distinct left cosets of H in G, then there exists a homomorphism $\psi : G \to A(S)$ such that $\ker \psi \subseteq H$.

(ii) Find a non-cyclic subgroup of order 12 (with justification) of the group $\mathbb{Z}_{40} \times \mathbb{Z}_{30}$.

(iii) Justify whether the following statement is true or not : "A group of order 250 must contain a subgroup of order 25." (4+2)+2+2

(d) (i) State generalised Cayley's theorem. Using this theorem show that a group G of order 80 containing a subgroup of order 16 is not a simple group.

(ii) Let G be a finite group and G acts on a finite non-empty set X. Then the number of orbits of G on X is $\frac{1}{|G|}\Sigma_{g \in G} F(g)$, where $F(g)$ is the number of elements of X fixed by g. 5+5

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