B.Sc./5th Sem (H)/MATH/23(CBCS)

2023

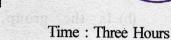
5th Semester Examination

MATHEMATICS (Honours)

Paper: C 12-T

[Group Theory II]

[CBCS]



Full Marks: 60

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any ten questions:

- 2×10=20
- (a) Prove that the automorphism group of $(\mathbb{Z}_8, +)$ is the Klein's Four Group.
- (b) Prove that the mapping $f:U_{16} \to U_{16}$ sending $x \mapsto x^3$ is an automorphism.
- (c) Let G_1 and G_2 be two groups. Then show that $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$.

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(2)

- (d) Show that $n\mathbb{Z}$ is a characteristic subgroup of $(\mathbb{Z}, +)$ for any positive integer n.
- (e) Compute the commutator subgroup of S_3
- (f) Is the direct product $\mathbb{Z} \times \mathbb{Z}$ a cyclic group? Justify your answer.
- (g) For any integer n > 1, give examples of two non-isomorphic groups of order n^2 .
- (h) Is the group $\mathbb{Z}_{75} \oplus \mathbb{Z}_{20}$ isomorphic to $\mathbb{Z}_{25} \oplus \mathbb{Z}_{60}$? Justify your answer.
- (i) Consider the left action of the group $G = GL(2, \mathbb{R})$ on $G = GL(2, \mathbb{R})$ by conjugation.

Find the orbit of $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

- (j) Give an example (with proper justification) of a group action on a group G so that the kernel of the action is the centre of G.
- (k) Let H be a subgroup of order 13 and index 5 of a group G. Prove that H is normal in G.
- (1) State Sylow's first theorem.
- (m) Using Sylow's Theorem, prove that a group of order 88 has a normal subgroup of order 11.



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- (n) Does there exist a group whose class equation is 10 = 1 + 1 + 1 + 2 + 5? Give reason in support of your answer.
- (o) Prove that any p-group has non-trivial centre.

Group - B

2. Answer any *four* questions :

5×4=20

- (a) Prove that the automorphism group of $(\mathbb{Q}, +)$ is isomorphic with (\mathbb{Q}^*, \cdot) .
- (b) Find out all the conjugate classes of S_4 .
- '(c) Let G be a group of order 2m, where m is an odd integer. Show that G has a normal subgroup of order m.
- (d) Show that the mapping $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by f((a,b)) = a-b is a group homomorphism. Find the kernel of f.
- (e) Let G be a group of order pq, where p, q are primes such that p > q and q does not divide p-1. Then show that G is a cyclic group.
- (f) Prove that a group of order 96 has a normal subgroup of order 16 or 32.

Group - C

3. Answer any two questions:

10×2=20

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- (a) (i) Compute the automorphism group of S_3 .
- (ii) Justify whether the following statement is true or not — " $\mathbb{Z}_4 \times \mathbb{Z}_6$ is isomorphic with \mathbb{Z}_{24} ".
- (iii) Find the number of elements of order 7 in a group of order 14.
- 3 (i) Let H and K be two subgroups of a group G such that $H \subseteq K$. Then prove that
- (1) if H is a characteristic subgroup of K and H is a characteristic subgroup of G, and K is a characteristic subgroup of G then
- (2) if H is a characteristic subgroup of K and a normal subgroup of G. K is a normal subgroup of G then H is
- (ii) Consider the action of the permutation group defined by $(\sigma, a) \mapsto \sigma(a)$ for all $\sigma \in G$ and (132)(45)} on the set $I_5 = \{1, 2, 3, 4, 5\}$ $G = \{(1), (123), (132), (45), (123)(45),$ for all $a \in I_5$. Then find the stabilizers of 1
- (iii) Is conjugate of a Sylow p-subgroup of a group G again a Sylow p-subgroup of G? Justify your answer. (3+3)+2+2

- (i) Let G be a group acting on a set S. Then H in G, then there exists a homomorphism $\psi:G\to A(S)$ such that ker $\psi\subseteq H$. and S be the set of all distinct left cosets of prove that if H is a subgroup of a group Gdenotes the permutation group of S. Hence homomorphism from G to A(S) where A(S)show that this group action induces a
- (ii) Find a non-cyclic subgroup of order 12 (with justification) of the group $\mathbb{Z}_{40} \times \mathbb{Z}_{30}$.
- (iii) Justify whether the following statement is true a subgroup of order 25." or not: "A group of order 250 must contain (4+2)+2+2
- <u>a</u> (i) State generalised Cayley's theorem. Using this simple group. containing a subgroup of order 16 is not a theorem show that a group G of order 80
- (ii) Let G be a finite group and G acts on a finite is the number of elements of X fixed by g. of G on X is $\frac{1}{|G|} \sum_{g \in G} F(g)$, where F(g)non-empty set X. Then the number of orbits