B.Sc./5th Sem (H)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (Honours)

Paper : C 11-T

[Partial Differential Equations and Applications]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any ***ten*** questions : 2×10=20

(a) Find the partial differential equation by eliminating arbitrary constants a and b from $u = ae^{-b^2t}\sin bx$.

(b) Verify that $u(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$ is a solution of $u_t = ku_{xx}$.

(c) Find the complete integral of the partial differential equation

$$\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right) = 1.$$

P.T.O.

(d) Sketch the characteristic curves of Hyperbolic and parabolic partial differential equations and mark the 'domain of dependence' and 'range of influence'.

(e) Show that the following PDEs $px - qy = x$, $px^2 + q = xz$ $\left(\text{where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}\right)$ are compatible.

(f) Find the characteristic curves of the PDE

$$\frac{\partial^2 z}{\partial y^2} - y\frac{\partial^2 z}{\partial x^2} = 0.$$

(g) Solve : $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$

(h) Find the particular solution of the PDE :

$\left(D^2 - 2DD' + D'^2\right)z = xy$ where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

(i) Propose a problem of semi-infinite string with (a) free end and (b) fixed end.

(j) A particle describes a curve $s = c\tan\psi$ with uniform speed v. Find the acceleration indicating its direction.

(k) A particle describes a curve whose equation is $\frac{a}{r} = \theta^2 + b$ under a force to the pole. Find the law of force.

(l) A particle moves in an ellipse under the force μ/r^2 towards the focus. Find the velocity at any point of its path.

(m) Let $a, b \in \mathbb{R}$ be such that $a^2 + b^2 \neq 0$. Then prove that the Cauchy problem $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$, $x, y \in \mathbb{R}$ with $z(x, y) = x$ on $ax + by = 1$ has a unique solution.

(n) Show that all the surfaces of revolution, $z = f\left(x^2 + y^2\right)$ with the z-axis as the axis of symmetry, where f is an arbitrary function, satisfy the partial differential equation $yp - xq = 0$.

(o) Find the nature of the PDE : $\left(x^2 - y^2\right)\frac{\partial^2 z}{\partial x^2} +$

$$2\left(x^2 + y^2\right)\frac{\partial^2 z}{\partial x \partial y} + \left(x^2 - y^2\right)\frac{\partial^2 z}{\partial y^2} = 0 \quad \text{for } x > 0$$

and $y > 0$.

P.T.O.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Solve the equation $y^2u_x^2 + x^2u_y^2 = (xyu)^2$ by separation of variable method.

(b) Solve $yz\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = xy$ and hence find the integral surface passing through $z^2 - y^2 = 1$, $x^2 - y^2 = 4$.

(c) A particle moves describing an ellipse under a force to the centre. if v, v_1, v_2 are the velocities at the ends of the latus rectum and major and minor axes respectively, prove that $v^2v_2^2 = v_1^2\left(2v_2^2 - v_1^2\right)$.

(d) Find the characteristics and reduce the equation to its canonical form

$$u_{xx} - 4u_{xy} + 4u_{yy} = e^y.$$

(e) If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the Sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.

(f) A particle moves in a plane under a force towards a fixed centre, proportional to the distance. If the

path of the particle has two apsidal distance a, b $(a > b)$, show that its equation can be written in the form $u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.

(ii) A machine gun of mass M_0 stands on a horizontal plane and contains a shot of total mass m, which is fixed horizontally at a uniform rate with constant velocity u relative to the gun. If the coefficient of sliding friction between the gun and the plane is μ, and the shot is all expended in time T, show that the velocity of the gun is then $u\log(1 + m/M_0) - \mu gT$. 4+6

(b) (i) Derive the Laplace equation for the distribution of gravitational potential.

(ii) Solve :

$$x\left(y^2 - z^2\right)p - y\left(z^2 + x^2\right)q = z\left(x^2 + y^2\right).$$

5+5

P.T.O.

(c) (i) Find the solution of the initial boundary value problem

$$u_t = ku_{xx},\ 0 < x < l,\ t > 0,$$
$$u(0, t) = 0,\ t \geq 0$$
$$u(l, t) = 0,\ t \geq 0,$$

$u(x, 0) = f(x),\ 0 \leq x \leq l$ using separation of variables.

(ii) If a particle moves in a plane curve, prove that the sum of kinetic energy and the potential energy is constant when the force is conservative. 7+3

(d) (i) If the orbit described by a particle under a central force to the origin be $r^n \cos n\theta = a^n$. Find the law of force.

(ii) Let $z(x, y)$ be the solution of the first order partial differential equation $x\frac{\partial z}{\partial x} + \left(x^2 + y\right)\frac{\partial z}{\partial y} = z,\ \forall x, y \in \mathbb{R}$ satisfying $z(2, y) = y - q$ $y \in \mathbb{R}$. Then find the value of $z(1, 2)$. 6+4
