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B.Sc./4th Sem (H)/MATH/23(CBCS)

2023

4th Semester Examination

MATHEMATICS (Honours)

Paper : C 10-T

[Ring Theory and Linear Algebra - I]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

1. Answer any *ten* questions : 2×10=20

(a) Define divisors of zero in a ring. Show that the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in R\right\}$ contains divisor of zero.

(b) Find the maximal ideals of Z_6, the ring of integers modulo 6.

(c) Prove that a ring R is commutative if and only if $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$.

P.T.O.

(d) If a is a fixed element of a ring R, show that $I_a = \{x \in R : ax = 0\}$ is a subring of R.

(e) Let $R = (Z, +, .)$, $R' = (3Z, +, .)$ and a mapping $\varphi : R \to R'$ be defined by $\varphi(a) = 3a, a \in R$. Examine if φ is a homomorphism.

(f) Let $(R, +, .)$ and $(R', +, .)$ be two rings and $f : R \to R'$ be a homomorphism. Then prove that $f(-x) = -f(x), x \in R$.

(g) Show that the solutions of differential equation $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$ is a subspace of the vector space of all real valued continuous function.

(h) Determine k so that the set S is linearly dependent in R^3, where $S = \{(k, 1, 1), (1, k, 1), (1, 1, k)\}$.

(i) Find a basis of the subspace W of R^3, where $W = \{(x, y, z) \in R^3 : 3x - 2y - z = 0\}$.

(j) Let V be a vector space over the field F and W_1, W_2 be two subspaces of V. Is $W_1 \cup W_2$ a subspace of V?

(k) Find a basis for the vector space R^3 that contains the vector (1, 2, 0).

(l) Let $T : R^2 \to R$ be define by $T(x, y) = |x - y|$. Examine that T is a linear transformation or not.

(m) Define rank and nullity of a linear transformation.

(n) Find a linear transformation $T : R^2 \to R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$.

(o) Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear transformation. If $\ker T = \{\theta\}$, then show that T is injective.

2. Answer any ***four*** questions : 5×4=20

(a) Prove that any non-trivial finite ring without zero divisors is a division ring.

(b) Find all maximal ideals of the ring $(\mathbb{Z}_8, +, \cdot)$.

(c) State and prove third isomorphism theorem for rings. 1+4

(d) Let V be a real vector space and $\alpha_1, \alpha_2, \alpha_3 \in V$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 = \theta$. Then prove that $W_1 = W_2 = W_3$ where W_1 is the subspace spanned by $\{\alpha_1, \alpha_2\}$, W_2 is the subspace spanned by $\{\alpha_2, \alpha_3\}$ and W_3 is the subspace spanned by $\{\alpha_3, \alpha_1\}$.

P.T.O.

(e) Let V be a finite dimensional vector space over the field F and W be a subspace of V. Then show that $\dim V/W = \dim V - \dim W$.

(f) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find T^{-1}.

3. Answer any ***two*** questions : 10×2=20

(a) (i) Show that $(R, +, \circ)$ does not form a ring where R is the set of all real valued continuous functions defined on the real line and '+' denotes the pointwise addition of functions and '$\circ$' denotes the composition of mappings.

(ii) Find all the zero divisors in the ring $\mathbb{Z}_4 \times \mathbb{Z}_6$.

(iii) Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$. Let

$$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$ and

$$W_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}.$$ Find a basis for the subspace $W_1 \cap W_2$. 4+3+3

(b) (i) Show that $12\mathbb{Z}$ is an ideal of the ring $(3\mathbb{Z}, +, \cdot)$.

(ii) Find all the ideals of the quotient ring $3\mathbb{Z}/12\mathbb{Z}$.

(iii) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

$$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$$

for all $(x, y, z) \in \mathbb{R}^3$. Let B be the standard ordered basis of $\mathbb{R}^3$. If $B_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis of $\mathbb{R}^3$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$, $\alpha_3 = (2, 1, 1)$, then find an invertible matrix P such that $[T]_{B_1} = P^{-1}[T]_B P$ where $[T]_{B_1}$ stands for the matrix representation of T with respect to B_1. 3+2+5

(c) (i) Let R be a commutative ring with 1. Then prove that R is an integral domain if and only if the zero ideal $\{0\}$ is a prime ideal of R.

(ii) Let V be a vector spaces over the field F with $\dim V = n$. Prove that V is isomorphic with the vector space F^n over F.

(iii) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V such that $rank\ T = rank\ T^2$. Then prove that $\ker T \cap \operatorname{Im} T = \{\theta\}$. 3+4+3

(d) (i) Give an example of an infinite ring whose characteristic is finite.

(ii) Let $f: \mathbb{Q} \to S$ and $g: \mathbb{Q} \to S$ be two ring homomorphisms from the field of rationals to a ring $(S, +, \cdot)$. Suppose $f(x) = g(x)$ for all $x \in \mathbb{Z}$. Then prove that $f = g$.

(iii) Find two linear operators T, U on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ such that UT is the zero operator on V but TU is not the zero operator on V.

(iv) Let T be the linear operator on $\mathbb{R}^2$ defined by $T(a, b) = (-b, a)$ for all $(a, b) \in \mathbb{R}^2$. Find the matrix representation of T with respect to the standard ordered basis of $\mathbb{R}^2$.

2+3+3+2
