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PG (CBCS)
M.SC. Semester-IV Examination, 2023
MATHEMATICS
PAPER: MTM 401
(FUNCTIONAL ANALYSIS)

Full Marks: 50 **Time: 2 Hours**

The figures in the right-hand margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer any **FOUR** questions from the following: 4×2=8

a) Let T be a linear transformation between two normed spaces. Prove that if T is continuous at 0 then T is continuous at all points.

b) Let $T \in B(H,Y)$ where H is a Hilbert space and Y is an inner product space. Prove that $||T^*||^2 = |(|T^* T|)| = |(|TT^*|)| = ||T||^2$.

c) Let H be a complex Hilbert space and $A \in BL(H), \langle Ax, x\rangle = 0, \forall x \in H$. Prove that $A = 0$.

d) State with justification, whether the following statement is true or false:
Let Y be a proper dense subspace of a Banach space X. Then Y is not a Banach space with respect to the induced norm.

e) Let X be a normed space and Y be a closed subset of X. If $x_n \xrightarrow{w} x$ in X, then show that$x_n + Y \xrightarrow{w} x + Y$ *in* X/Y.

f) Let S be a non-empty subset of an inner product space X. Show that $S^{\perp}$ is a closed linear subspace of X.

2. Answer any **FOUR** questions from the following: 4×4=16

a) Suppose that X and Y are Banach spaces, $T: X \to Y$ is a linear map which has a closed graph. Prove that T is continuous.

b) (i) Define adjoint operator.
(ii) Let $T \in B(H,Y)$ where H is a Hilbert space and Y is an inner product space. Prove that the adjoint T^* of T is the unique mapping of Y into H such that $\langle Tx, y\rangle = \langle x, T^* y\rangle, \forall x \in H$ and $y \in Y$.

c) Suppose X and Y are Banach spaces and $T \in B(X,Y)$ is injective. If T is bounded below then prove that $Rang(T)$ is closed.

d) If $T \in B(H)$ is such that $< Tx, x > = 0$ for all $x \in H$, then show that T=0, where H is a complex Hilbert space.

e) Let $X = \mathbb{C}^3$. *For* $x = (x(1), x(2), x(3)) \in X$, let $||x|| = \left[(|x(1)|^2 + |x(2)|^2)^{\frac{3}{2}} + |x(3)|^3\right]^{\frac{1}{3}}$. Show that $||.||$ is a norm on X. **(P.T.O)**

f) Let X and Y be inner product spaces. Then show that a linear map $F: X \to Y$ satisfies $\langle (x), F(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$ if and only if it satisfies $\| F(x) \| = \| x \|$ for all $x \in X$ where the norms on X and Y are induced by the respective inner products.

3. Answer any **TWO** questions from the following: 2×8=16

a) (i) Prove that a Banach space cannot have a countable infinite basis.

(ii) Let X and Y be Banach spaces and $T \in B(X, Y)$. If T is bijective, prove that there exists $S \in B(X, Y)$ such that $ST = id_X$ and $TS = id_Y$. 4 + 4

b) Let, $\{u_1, u_2, u_3, \ldots\}$ be an orthogonal set in an inner product space X and let $k_1, k_2, k_3, \ldots \in \mathbb{C}$.

(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then prove that $k_n = \langle x, u_n\rangle$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$.

(ii)If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, prove that $\sum_{n=1}^{\infty} k_n u_n$ converges in X. 4 + 4

c) (i) Let M be a closed subspace of a Hilbert space H and $x \in H$. Then show that there exist unique $y \in M$ and $z \in M^{\perp}$ such that $x = y + z$.

(ii) Show that a normed space X is a Banach space if and only if every absolutely summable series of elements of X is summable in X. 3 + 5

d) (i) Let X be an inner product space and $A, B \subseteq X$. Then show that

(a) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$, (b) $A \subseteq A^{\perp\perp}$, (c) $A^{\perp} = A^{\perp\perp\perp}$.

(ii) If $||x + \lambda y|| = ||x - \lambda y||$ is true for all scalar λ, then show that $x \perp y$.

6 + 2

[Internal Assessment- 10 Marks]

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