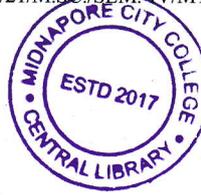


PG (CBCS)
M.Sc. Semester-IV Examination, 2023
MATHEMATICS
PAPER: MTM 401
(FUNCTIONAL ANALYSIS)



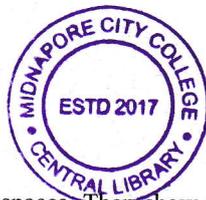
Full Marks: 50

Time: 2 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer any **FOUR** questions from the following: 4×2=8
- a) Let T be a linear transformation between two normed spaces. Prove that if T is continuous at 0 then T is continuous at all points.
- b) Let $T \in B(H, Y)$ where H is a Hilbert space and Y is an inner product space. Prove that $\|T^*\|^2 = |(T^* T)| = |(T T^*)| = \|T\|^2$.
- c) Let H be a complex Hilbert space and $A \in BL(H)$, $\langle Ax, x \rangle = 0, \forall x \in H$. Prove that $A = 0$.
- d) State with justification, whether the following statement is true or false:
Let Y be a proper dense subspace of a Banach space X . Then Y is not a Banach space with respect to the induced norm.
- e) Let X be a normed space and Y be a closed subset of X . If $x_n \xrightarrow{w} x$ in X , then show that $x_n + Y \xrightarrow{w} x + Y$ in X/Y .
- f) Let S be a non-empty subset of an inner product space X . Show that S^\perp is a closed linear subspace of X .
2. Answer any **FOUR** questions from the following: 4×4=16
- a) Suppose that X and Y are Banach spaces, $T: X \rightarrow Y$ is a linear map which has a closed graph. Prove that T is continuous.
- b) (i) Define adjoint operator.
(ii) Let $T \in B(H, Y)$ where H is a Hilbert space and Y is an inner product space. Prove that the adjoint T^* of T is the unique mapping of Y into H such that $\langle Tx, y \rangle = \langle x, T^* y \rangle, \forall x \in H$ and $y \in Y$.
- c) Suppose X and Y are Banach spaces and $T \in B(X, Y)$ is injective. If T is bounded below then prove that $Rang(T)$ is closed.
- d) If $T \in B(H)$ is such that $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T=0$, where H is a complex Hilbert space.
- e) Let $X = \mathbb{C}^3$. For $x = (x(1), x(2), x(3)) \in X$, let $\|x\| = [(|x(1)|)^2 + |x(2)|^2]^{\frac{3}{2}} + |x(3)|^3]^{\frac{1}{3}}$. Show that $\|\cdot\|$ is a norm on X . (P.T.O)



f) Let X and Y be inner product spaces. Then show that a linear map $F: X \rightarrow Y$ satisfies $\langle (x), F(y) \rangle = \langle x, y \rangle$ for all $x, y \in X$ if and only if it satisfies $\|F(x)\| = \|x\|$ for all $x \in X$ where the norms on X and Y are induced by the respective inner products.

3. Answer any **TWO** questions from the following: $2 \times 8 = 16$

- a) (i) Prove that a Banach space cannot have a countable infinite basis. $2 \times 8 = 16$
(ii) Let X and Y be Banach spaces and $T \in B(X, Y)$. If T is bijective, prove that there exists $S \in B(X, Y)$ such that $ST = id_X$ and $TS = id_Y$. $4 + 4$
- b) Let, $\{u_1, u_2, u_3, \dots\}$ be an orthogonal set in an inner product space X and let $k_1, k_2, k_3, \dots \in \mathbb{C}$.
(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then prove that $k_n = \langle x, u_n \rangle$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$.
(ii) If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, prove that $\sum_{n=1}^{\infty} k_n u_n$ converges in X . $4 + 4$
- c) (i) Let M be a closed subspace of a Hilbert space H and $x \in H$. Then show that there exist unique $y \in M$ and $z \in M^\perp$ such that $x = y + z$.
(ii) Show that a normed space X is a Banach space if and only if every absolutely summable series of elements of X is summable in X . $3 + 5$
- d) (i) Let X be an inner product space and $A, B \subseteq X$. Then show that
(a) $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$, (b) $A \subseteq A^{\perp\perp}$, (c) $A^\perp = A^{\perp\perp\perp}$.
(ii) If $\|x + \lambda y\| = \|x - \lambda y\|$ is true for all scalar λ , then show that $x \perp y$. $6 + 2$

[Internal Assessment- 10 Marks]
