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PG (CBCS)
M.SC Semester- II Examination, 2023
MATHEMATICS
PAPER: MTM 203
(ABSTRACT ALGEBRA & LINEAR ALGEBRA)

Full Marks: 50 **Time: 2 Hours**

Write the answer for each unit in separate sheet

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

MTM 203.1

ABSTRACT ALGEBRA

1. Answer any **TWO** questions from the following: 2×2=04

   a) Let $G = (\mathbb{R}, +), H = (\mathbb{Z}, +)$ and $G' = (\{z \in \mathbb{C}: |z| = 1\}, .)$. Prove that $\frac{G}{H} \cong G'$.
   b) Show that every finite field extension is an algebraic extension.
   c) Define Noetherian ring.
   d) Determine the degree of $[\mathbb{Q}(\sqrt{3}+2\sqrt{2}): \mathbb{Q}]$.

2. Answer any **TWO** questions from the following: 2×4=08

   a) Let I be an ideal of the Noetherian ring R. Prove the followings:
      i) The quotient R/I is a Noetherian ring.
      ii) Every nonempty set of ideals of R contains a maximal element under inclusion. 2+2
   b) Prove that in a commutative integral domain every prime element is irreducible. But not conversely. 4
   c) If $K \subseteq F \subseteq L$ is a tower of fields then show that
      $$[L: F][F: K] = [L: K]$$
      where $[L:F]$ denotes the degree of L over F.
   d) Let $F \subseteq K, K'$ be two field extensions of F. Let $\psi: K \to K'$ be an F- isomorphism. Let $\alpha \in K$ be a root of $f(x) \in F[x]$. Then $\psi(\alpha)$ is a root of $f(x)$.

3. Answer any **ONE** questions from the following: 1×8=08

   a) Define Artinian ring with an appropriate example. Prove that there are only finitely many maximal ideals in an Artinian ring. 3+5
   b)
      i) Let E be a field and G a finite group of automorphisms of E. Then show that E/E^G is a finite Galois extension. 5
      ii) Show that the Galois group of the Galois extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is a cyclic group of order n. 3

[Internal Assessment -05 Marks]

(P.T.O)

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MTM 203.2

LINEAR ALGEBRA

4. Answer any **TWO** questions from the following: 2×2=04

a) Let V be the vector space of real continuous functions on the interval $-\pi \leq t \leq \pi$ with inner product defined by $<f,g>= \int_{-\pi}^{\pi} f(t)g(t)dt$. Then $S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots.\}$ is orthogonal, orthonormal?

b) Is the operator $T(z,w) = (-w,z)$ on $\mathbb{C}^2$ self- adjoint? Justify your answer.

c) Justify the statements as true or false
   (i) Every linear operator has an adjoint.
   (ii) The adjoint of a linear operator is unique.

d) Show that similar matrices have the same minimal polynomial.

5. Answer any **TWO** questions from the following: 2×4=08

a) Let V be the vector space of all polynomial functions p from R into R which have degree 2 or less. Define three functions on V given by $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$, $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ is a basis of V^*(dual basis). Determine a basis for V such that $\{f_1, f_2, f_3\}$ is its dual basis.

b) Find all possible Jordan canonical forms for a linear operator $T: V\ to\ V$(vector space) where characteristic polynomial is $(t-2)^3(t-5)^5$. In each case, find the minimal polynomial $m(t)$.

c) Suppose that T is a normal operator on V and that 3 and 4 are eigen values of T. Prove that there exists a vector $v \in V$ such that $||v|| = \sqrt{2}$ and $||Tv|| = 5$.

d) Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Is A nilpotent? If yes, what is its index? Find the canonical form of A. 1+1+2

6. Answer any **ONE** questions from the following: 1×8=8

a) Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to canonical form through an orthogonal transformation. Find the nature, rank, index and signature of it. 4+1+1+1+1

b) 3+3+2
   i) Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if TU=UT.
   ii) Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigen vectors of T. Hence prove that T is self-adjoint.
   iii) If T is normal and $T^3 = T^2$, show that T is idempotent. If normality of T is dropped, does the conclusion still true?

[Internal Assessment- 05 Marks]

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