

PG (CBCS)
M.SC Semester- II Examination, 2023
MATHEMATICS
PAPER: MTM 203
(ABSTRACT ALGEBRA & LINEAR ALGEBRA)

Full Marks: 50

Time: 2 Hours

Write the answer for each unit in separate sheet

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

MTM 203.1

ABSTRACT ALGEBRA

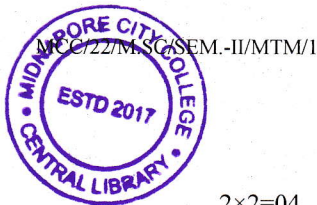
1. Answer any **TWO** questions from the following: 2×2=04
 - a) Let $G = (\mathbb{R}, +)$, $H = (\mathbb{Z}, +)$ and $G' = (\{z \in \mathbb{C} : |z| = 1\}, \cdot)$. Prove that $\frac{G}{H} \cong G'$.
 - b) Show that every finite field extension is an algebraic extension.
 - c) Define Noetherian ring.
 - d) Determine the degree of $[\mathbb{Q}(\sqrt{3+2\sqrt{2}}) : \mathbb{Q}]$.

2. Answer any **TWO** questions from the following: 2×4=08
 - a) Let I be an ideal of the Noetherian ring R . Prove the followings:
 - i) The quotient R/I is a Noetherian ring.
 - ii) Every nonempty set of ideals of R contains a maximal element under inclusion. 2+2
 - b) Prove that in a commutative integral domain every prime element is irreducible. But not conversely. 4
 - c) If $K \subseteq F \subseteq L$ is a tower of fields then show that $[L : F][F : K] = [L : K]$ where $[L : F]$ denotes the degree of L over F .
 - d) Let $F \subseteq K, K'$ be two field extensions of F . Let $\psi: K \rightarrow K'$ be an F - isomorphism. Let $\alpha \in K$ be a root of $f(x) \in F[x]$. Then $\psi(\alpha)$ is a root of $f(x)$.

3. Answer any **ONE** questions from the following: 1×8=08
 - a) Define Artinian ring with an appropriate example. Prove that there are only finitely many maximal ideals in an Artinian ring. 3+5
 - b)
 - i) Let E be a field and G a finite group of automorphisms of E . Then show that E/E^G is a finite Galois extension. 5
 - ii) Show that the Galois group of the Galois extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is a cyclic group of order n . 3

[Internal Assessment -05 Marks]

(P.T.O)



(2)
MTM 203.2
LINEAR ALGEBRA

4. Answer any **TWO** questions from the following: 2×2=04
- Let V be the vector space of real continuous functions on the interval $-\pi \leq t \leq \pi$ with inner product defined by $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$. Then $S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \dots\}$ is orthogonal, orthonormal?
 - Is the operator $T(z, w) = (-w, z)$ on \mathbb{C}^2 self-adjoint? Justify your answer.
 - Justify the statements as true or false
 - Every linear operator has an adjoint.
 - The adjoint of a linear operator is unique.
 - Show that similar matrices have the same minimal polynomial.

5. Answer any **TWO** questions from the following: 2×4=08
- Let V be the vector space of all polynomial functions p from R into R which have degree 2 or less. Define three functions on V given by $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$, $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ is a basis of V^* (dual basis). Determine a basis for V such that $\{f_1, f_2, f_3\}$ is its dual basis.
 - Find all possible Jordan canonical forms for a linear operator $T: V$ to V (vector space) where characteristic polynomial is $(t - 2)^3(t - 5)^5$. In each case, find the minimal polynomial $m(t)$.
 - Suppose that T is a normal operator on V and that 3 and 4 are eigen values of T . Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.
 - Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Is A nilpotent? If yes, what is its index? Find the canonical form of A . 1+1+2

6. Answer any **ONE** questions from the following: 1×8=8
- Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to canonical form through an orthogonal transformation. Find the nature, rank, index and signature of it. 4+1+1+1+1
 - Let T and U be self-adjoint operators on an inner product space V . Prove that TU is self-adjoint if and only if $TU=UT$. 3+3+2
 - Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigen vectors of T . Hence prove that T is self-adjoint.
 - If T is normal and $T^3 = T^2$, show that T is idempotent. If normality of T is dropped, does the conclusion still true?

[Internal Assessment- 05 Marks]
