Total pages: 02

PG CBCS
M.SC. Semester- I Examination, 2023
MATHEMATICS
PAPER: MTM 103
(ORDINARY DIFFERENTIAL EQUATIONS & SPECIAL FUNCTIONS)

Full Marks: 50 **Time: 2 Hours**

The figures in the right-hand margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

GROUP-A

1. Answer any FOUR of the following questions: **2×4=8**

a) Prove that $J_{-n}(z) = (-1)^n J_n(z)$ for integral n.

b) Define Green's function of the differential operator **L** of the non-homogeneous differential equation: $Lu(x) = f(x)$.

c) Show that $\int_{-1}^{1} P_n(z)dz = \begin{cases} 0, & n \neq 0 \\ 2, & n = 0 \end{cases}$ where the symbols have usual meaning.

d) Find all the singularities of the following differential equation and then classify them:

$$(z - z^2)w'' + (1 - 5z)w' - 4w = 0.$$

e) Define orthogonal functions associated with Strum-Liouville problem.

f) Prove that: $F(-n; b, b; -z) = (1 + z)^n$ where $F(a;\ b, c; z)$ denotes the hypergeometric function.

GROUP-B

2. Answer any FOUR of the following questions: **4×4=16**

a) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| < 1$, and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \geq 1$.

b) Show that $1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \cdots + (2n + 1)P_n(z) = \frac{d}{dz}[P_{n+1}(z) + P_n(z)]$, where $P_n(z)$ denotes the Legendre's Polynomial of degree n.

c) Consider the boundary value problem $\frac{d^2y}{dx^2} + \lambda y = 0, 0 \leq x \leq \pi$ subject to $y(0) = 0, y'(\pi) = 0$. Find the eigen values and eigen functions of the problem.

d) Using Green's function method, solve the following differential equation $y'' - y = x$, subject to boundary conditions $y(0) = y(1) = 0$.

(P.T.O.)

e) Prove that $\int_{-1}^{1} P_m(z)P_n(z)dz = \frac{2}{2n+1}\delta_{mn}$, where δ_{mn} and $P_n(z)$ are the Kroneker delta and Legendre's polynomial respectively.

f) Show that $\sqrt{\frac{\pi z}{2}} J_{\frac{3}{2}}(z) = \frac{1}{z} sinz - cosz$.

GROUP-C

3. Answer any TWO of the following questions: **2×8=16**

a) (i) Find the general solution of the homogeneous system $\frac{dX}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(ii) Show that $nP_n(z) = zP_n'(z) - P_{n-1}'(z)$, where $P_n(z)$ denotes the Legendre's Polynomial of degree n. 6+2

b) (i) Derive the Rodrigue's formula for Legendre's polynomial

(ii) Show that $(n+1)P_{n+1}(z) + nP_{n-1}(z) = (2n+1)zP_n(Z)$.
4+4

c) (i) Prove that if $f(z)$ is continuous and has continuous derivatives in $[-1,1]$ then $f(z)$ has unique Legendre's series expansion is given by $f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$ where P_n's are Legendre Polynomials $C_n = \frac{2n+1}{2}\int_{-1}^{1} f(z)P_n(z), n = 1,2,3, ...$

(ii) Prove that $\frac{d}{dz}[J_0(z)] = -J_1(z)$. 6+2

d) (i) Find the general solution of the ODE $2zw''(z) + (1+z)w'(z) - kw = 0$. (where k is a real constant) in series form for which values of k is there a polynomial solution?

(ii) Deduce the integral formula for hypergeometric function. 5+3

[Internal Assessment- 10 Marks]

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