

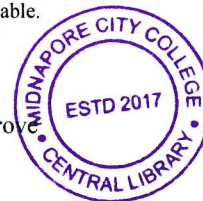
PG CBCS
M.SC. Semester-IV Examination, 2022
MATHEMATICS
 PAPER: MTM 401
 (FUNCTIONAL ANALYSIS)

Full Marks: 40**Time: 2 Hours**

The figures in the right-hand margin indicate full marks.
 Candidates are required to give their answers in their own words as far as practicable.

GROUP-A**1. Answer any four questions of the following: 4×2=8**

- a) Let T be a linear transformation between two normed spaces. Prove that if T is continuous at 0 then T is continuous at all points.
- b) State inverse mapping theorem.
- c) Let X be a normed space. Show that $x_n \rightarrow x$ weakly in X does not imply $x_n \rightarrow x$ in X in general.
- d) Show that every normed space can be embedded as a dense subspace of a Banach space.
- e) Let H be a Hilbert space and F be a closed subspace of H . Prove that $H = F + F^\perp$.
- f) Let $T \in BL(H)$ be self-adjoint. Show that $\text{Ker}(T) = \text{Ker}(T^*)$.

**GROUP-B****2. Answer any four questions of the following: 4×4=16**

- a) If $(V, \|\cdot\|)$ is a normed space and M is a finite dimensional subspace of V . Prove that M is closed.
- b) Show that $\langle Ae_j, e_i \rangle = (i+j+1)^{-1}$ for $0 \leq i, j \leq \infty$ defines a bounded operator on $l^2(\mathbb{N} \cup \{0\})$ with $\|A\| \leq \pi$.
- c) (i) Define best approximation from a set to a point.
 (ii) Let X be an inner product space and E be a convex subset of X . Prove that there exists at most one best approximation from E to any $x \in X$.
- d) (i) Define adjoint operator.

[P. T. O]

(ii) Let $T \in B(H, Y)$ where H is a Hilbert space and Y is an inner product space. Prove that the adjoint T^* of T is the unique mapping of Y into H such that $\langle Tx, y \rangle = \langle x, T^*y \rangle, \forall x \in H$ and $y \in Y$.

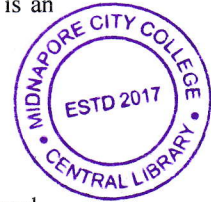
e) Let $S \in BL(H)$, where H is a Hilbert space. Prove that for all $x, y \in H, \langle Sx, y \rangle = \frac{1}{4} \sum_{n=0}^3 i^n \langle S(x + i^n y), (x + i^n y) \rangle$.

f) Give an example to show that the completeness of the domain is an essential requirement in the Uniform Boundedness Principle.

GROUP-C

3. Answer any two questions of the following:

8×2=16



a) (i) If X is a normed space, M is a closed subspace of $X, x_0 \in X \setminus M$ and $d = \text{dist}(x_0, M)$, show that there is an $f \in X^*$ such that $f(x_0) = 1, f(x) = 0$ for all $x \in M$ and $\|f\| = d^{-1}$. 5

(ii) Let S be the Unilateral shift operator. Show that $S^{*n} \xrightarrow{S} 0$ but not uniformly. 3

b) Let, $\{u_1, u_2, u_3, \dots\}$ be an orthogonal set in an inner product space X and let $k_1, k_2, k_3, \dots \in \mathbb{C}$.

(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then prove that $k_n = \langle x, u_n \rangle$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$. 4

(ii) If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, prove that $\sum_{n=1}^{\infty} k_n u_n$ converges in X . 4

c) (i) Let $S, T \in BL(H, Y)$ and $\alpha \in \mathbb{C}$, prove that $(S + T)^* = S^* + T^*$ and $(\alpha T)^* = \bar{\alpha} T^*$. 4

(ii) Let H, K be two Hilbert spaces and Y be an inner product space. If $T \in BL(H, K)$ and $S \in BL(K, Y)$, prove that $(ST)^* = T^* S^*$. 4
