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PG CBCS
M.SC. Semester-IV Examination, 2022
MATHEMATICS
PAPER: MTM 401

(FUNCTIONAL ANALYSIS)

Full Marks: 40 **Time: 2 Hours**

The figures in the right-hand margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

GROUP-A

1. Answer any four questions of the following: **4×2=8**

a) Let T be a linear transformation between two normed spaces. Prove that if T is continuous at 0 then T is continuous at all points.

b) State inverse mapping theorem.

c) Let X be a normed space. Show that $x_n \to x$ weakly in X does not imply $x_n \to x$ in X in general.

d) Show that every normed space can be embedded as a dense subspace of a Banach space.

e) Let H be a Hilbert space and F be a closed subspace of H. Prove that $H = F + F^{\perp}$.

f) Let $T \in BL(H)$ be self-adjoint. Show that Ker(T)=Ker(T*).

GROUP-B

2. Answer any four questions of the following: **4×4=16**

a) If $(V, ||.||)$ is a normed space and M is a finite dimensional subspace of V. Prove that M is closed.

b) Show that $< Ae_j, e_i >= (i + j + 1)^{-1}$ for $0 \leq i, j \leq \infty$ defines a bounded operator on $l^2(\mathbb{N} \cup \{0\})$ with $\| A \| \leq \pi$.

c) (i) Define best approximation from a set to a point.

(ii) Let X be an inner product space and E be a convex subset of X. Prove that there exists at most one best approximation from E to any $x \in X$.

d) (i) Define adjoint operator.

(ii) Let $T \in B(H,Y)$ where H is a Hilbert space and Y is an inner product space. Prove that the adjoint T^* of T is the unique mapping of Y into H such that $\langle Tx, y\rangle = \langle x, T^*y\rangle, \forall x \in H$ and $y \in Y$.

e) Let $S \in BL(H)$, where H is a Hilbert space. Prove that for all $x, y \in H, < Sx, y > = \frac{1}{4}\sum_{n=0}^{3} i^n < S(x + i^n y), (x + i^n y) >$.

f) Give an example to show that the completeness of the domain is an essential requirement in the Uniform Boundedness Principle.

GROUP-C

3. Answer any two questions of the following: **8×2=16**

a) (i) If X is a normed space, M is a closed subspace of X, $x_0 \in X \setminus M$ and $d = dist(x_0, M)$, show that there is an $f \in X^*$ such that $f(x_0)=1$, $f(x)=0$ for all $x \in M$ and $\|f\| = d^{-1}$. 5

(ii) Let S be the Unilateral shift operator. Show that $S^{*n} \xrightarrow{S} 0$ but not uniformly. 3

b) Let, $\{u_1, u_2, u_3, \ldots\}$ be an orthogonal set in an inner product space X and let $k_1, k_2, k_3, \ldots \in \mathbb{C}$.

(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then prove that $k_n = \langle x, u_n\rangle$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$. 4

(ii) If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, prove that $\sum_{n=1}^{\infty} k_n u_n$ converges in X. 4

c) (i) Let $S, T \in BL(H,Y)$ and $\alpha \in \mathbb{C}$, prove that $(S+T)^* = S^* + T^*$ and $(\alpha T)^* = \bar{\alpha} T^*$. 4

(ii) Let H, K be two Hilbert spaces and Y be an inner product space. If $T \in BL(H,K)$ and $S \in BL(K,Y)$, prove that $(ST)^* = T^* S^*$. 4

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