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PG CBCS

M.Sc. Semester-II Examination, 2022

(Mathematics)

PAPER: MTM 203

(ABSTRACT ALGEBRA AND LINEAR ALGEBRA)

Full Marks: 40 **Time: 2 Hours**

Write the answer for each unit in separate sheet

The figures in the right-hand margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

MTM 203.1: ABSTRACT ALGEBRA

Marks: 20

1. **Answer any two questions:** 2×2=4

 a) Let $G = (\mathbb{R}, +), H = (\mathbb{Z}, +)$ and $G' = (\{z \in \mathbb{C}: |z| = 1\}, .)$. Prove that $\frac{G}{H} \cong G'$.

 b) Prove that if an ideal S of a ring with unity R contains a unit of R, then $S = R$.

 c) An element x in a group G is called commutator if $x = aba^{-1}b^{-1}$ for some $a, b \in G$. By H we mean the subgroup generated by all the commutators. Show that G/H is abelian.

 d) Give an example of an infinite quotient group.

2. **Answer any two questions:** 2×4=8

 a) Define direct product of the groups $G_1, G_2, \dots, G_n$. Then show that the center of a direct product is the direct product of the centers, i.e. $Z(G_1 \times G_2 \times \dots \times G_n) = Z(G_1) \times Z(G_2) \times \dots \times Z(G_n)$.

 b) In a commutative ring R with unity, an ideal P is a prime ideal if and only if the quotient ring $\frac{R}{P}$ is an integral domain.

 c) Let $K \subseteq F$ be a field extension and $\alpha \in F$ be algebraic over K. Then show that there exists a unique monic irreducible polynomial $f(x) \in K[x]$ such that $f(\alpha) = 0$.

 d) If G is a group acting on a set S, and s is a fixed element of S then define stabilizer and kernel of the group action. Prove that stabilizer of the group action is a subgroup of G.

[P.T.O]

3. Answer any one questions: **1×8=8**

a) (i) Show that if R is a principal ideal domain, then it is a unique factorization domain. Give an example to show that the converse is not true.

(ii) If $K \subseteq F \subsetneq L$ is a tower of fields then show that $[L : F][F : K] = [L : K]$. Where $[L: F]$ denotes the degree of L over F. [4+4]

b) Define solvable group with an appropriate example. Prove that any group G of order p^n where p is a prime number is solvable. [3+5]

MTM 203.2: LINEAR ALGEBRA

Marks: 20

1. Answer any two questions: **2×2=4**

a) Let A be a real square matrix. Is A similar to a Jordan matrix? If not, give a counter example.

b) The linear map G: $\mathbf{R}^2 \to \mathbf{R}^3$ defined by $G(x,y)=(x+y,\ x-2y,\ 3x+y)$ is non-singular. Find G^{-1}.

c) Give an example of two self-adjoint transformations whose product is not self-adjoint.

d) If T is normal and $T^3 = T^2$, show that T is idempotent. If normality of T is dropped, does the conclusion still true?

2. Answer any two questions: **2×4=8**

a) A linear operator on $\mathbb{R}^2$ is defined by $T(x+y) = (x+2y, x-y)$. Find the adjoint, i.e., T^*if the inner product is standard one. If $\alpha = (1, 3)$, find $T^*(\alpha)$.

b) Let n be a (+)ve integer and V be an (n+1)-dimensional vector space over R. If $\{e_1, e_2, \ldots, e_{n+1}\}$ be a basis of V and $T: V \to V$ be the linear transformation satisfying $T(e_i) = e_{i+1}$ for i = 1,2,…,n and $T(e_{n+1}) = 0$. Then find rank T and trace of T.

c) Show that if A be the matrix representation of a bilinear form f then for u, v ∈ V, $f(u, v) = [u]^T A[v]$, where [u] denotes the coordinate vector of u in the given basis S.

[P.T.O]

d) Suppose T is a linear operator on an inner product space. Then T is normal if and only if its real and imaginary parts commute.

3. Answer any <u>one</u> questions: **1×8=8**

a) (i) Let P_2 be a family of polynomials of degree 2 at most. Define an inner product on P_2 as $< f(x), g(x) > = \int_0^1 f(x)g(x)dx$. Let $\{1, x, x^2\}$ be a basis of the inner product space P_2. Find out an orthonormal basis from the basis.

(ii) Prove that a necessary and sufficient condition that an $n \times n$ matrix A over F be diagonalizable is that A as n linearly independent eigen vectors in $V_n(F)$. **[4+4]**

b) What do you mean by an annihilator (W^0) of a subset W of a vector space V. Show that W^0 is a subspace of V* (where V* be the dual space of V). Find a basis of the annihilator W^0 of the subspace W of $\mathbf{R}^4$ spanned by $v_1 = (1, 2, -3, 4)$, $v_2 = (0, 1, 4, -1)$.

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