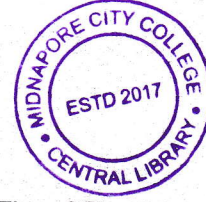


PG CBCS
M.Sc. Semester-II Examination, 2022
(Mathematics)
PAPER: MTM 203
(ABSTRACT ALGEBRA AND LINEAR ALGEBRA)

**Full Marks: 40****Time: 2 Hours****Write the answer for each unit in separate sheet**

The figures in the right-hand margin indicate full marks.
 Candidates are required to give their answers in their own words as far as practicable.

MTM 203.1: ABSTRACT ALGEBRA**Marks: 20**

1. **Answer any two questions:** **2×2=4**
- a) Let $G = (\mathbb{R}, +)$, $H = (\mathbb{Z}, +)$ and $G' = (\{z \in \mathbb{C} : |z| = 1\}, \cdot)$. Prove that $\frac{G}{H} \cong G'$.
- b) Prove that if an ideal S of a ring with unity R contains a unit of R , then $S = R$.
- c) An element x in a group G is called commutator if $x = aba^{-1}b^{-1}$ for some $a, b \in G$. By H we mean the subgroup generated by all the commutators. Show that G/H is abelian.
- d) Give an example of an infinite quotient group.
2. **Answer any two questions:** **2×4=8**
- a) Define direct product of the groups G_1, G_2, \dots, G_n . Then show that the center of a direct product is the direct product of the centers, i.e. $Z(G_1 \times G_2 \times \dots \times G_n) = Z(G_1) \times Z(G_2) \times \dots \times Z(G_n)$.
- b) In a commutative ring R with unity, an ideal P is a prime ideal if and only if the quotient ring $\frac{R}{P}$ is an integral domain.
- c) Let $K \subseteq F$ be a field extension and $\alpha \in F$ be algebraic over K . Then show that there exists a unique monic irreducible polynomial $f(x) \in K[x]$ such that $f(\alpha) = 0$.
- d) If G is a group acting on a set S , and s is a fixed element of S then define stabilizer and kernel of the group action. Prove that stabilizer of the group action is a subgroup of G .

[P.T.O]

[2]

3. Answer any one questions: 1×8=8
- a) (i) Show that if R is a principal ideal domain, then it is a unique factorization domain. Give an example to show that the converse is not true.
- (ii) If $K \subseteq F \subseteq L$ is a tower of fields then show that $[L : F][F : K] = [L : K]$. Where $[L : F]$ denotes the degree of L over F . [4+4]
- b) Define solvable group with an appropriate example. Prove that any group G of order p^n where p is a prime number is solvable. [3+5]

MTM 203.2: LINEAR ALGEBRA

Marks: 20

1. Answer any two questions: 2×2=4
- a) Let A be a real square matrix. Is A similar to a Jordan matrix? If not, give a counter example.
- b) The linear map $G: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $G(x,y) = (x+y, x-2y, 3x+y)$ is non-singular. Find G^{-1} .
- c) Give an example of two self-adjoint transformations whose product is not self-adjoint.
- d) If T is normal and $T^3 = T^2$, show that T is idempotent. If normality of T is dropped, does the conclusion still true?
2. Answer any two questions: 2×4=8
- a) A linear operator on \mathbb{R}^2 is defined by $T(x+y) = (x+2y, x-y)$. Find the adjoint, i.e., T^* if the inner product is standard one. If $\alpha = (1, 3)$, find $T^*(\alpha)$.
- b) Let n be a (+)ve integer and V be an $(n+1)$ -dimensional vector space over \mathbb{R} . If $\{e_1, e_2, \dots, e_{n+1}\}$ be a basis of V and $T: V \rightarrow V$ be the linear transformation satisfying $T(e_i) = e_{i+1}$ for $i = 1, 2, \dots, n$ and $T(e_{n+1}) = 0$. Then find rank T and trace of T .
- c) Show that if A be the matrix representation of a bilinear form f then for $u, v \in V$, $f(u, v) = [u]^T A [v]$, where $[u]$ denotes the coordinate vector of u in the given basis S .

[P.T.O]

[3]

d) Suppose T is a linear operator on an inner product space. Then T is normal if and only if its real and imaginary parts commute.

3. Answer any one questions:

1×8=8

a) (i) Let P_2 be a family of polynomials of degree 2 at most. Define an inner product on P_2 as $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx$. Let $\{1, x, x^2\}$ be a basis of the inner product space P_2 . Find out an orthonormal basis from the basis.

(ii) Prove that a necessary and sufficient condition that an $n \times n$ matrix A over F be diagonalizable is that A has n linearly independent eigen vectors in $V_n(F)$. [4+4]

b) What do you mean by an annihilator (W^0) of a subset W of a vector space V . Show that W^0 is a subspace of V^* (where V^* be the dual space of V). Find a basis of the annihilator W^0 of the subspace W of \mathbf{R}^4 spanned by $v_1 = (1, 2, -3, 4)$, $v_2 = (0, 1, 4, -1)$.

