Total Pages : 12 B.Sc./3rd Sem (H)/MATH/22(CBCS)

2022

3rd Semester Examination

MATHEMATICS (Honours)

Paper : GE 3-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

(Differential Equations & Vector Calculus)

1. Answer any ***ten*** questions from the following : 2×10=20

(a) If $y = x$ is a solution of the differential equation $\left(x^2D^2 + xD - 1\right)y = 0,\ D = \dfrac{d}{dx}$, then find it's second linearly independent solution.

(b) State the Principle of Superposition for homogeneous equation.

(c) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

P.T.O.

(d) The acceleration of a particle at any time $t \geq 0$ is given by $\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\hat{i} - 8\sin 2t\hat{j} + 16t\hat{k}$. If the velocity $\vec{v}$ be zero at $t = 0$, find $\vec{v}$ at any time.

(e) Determine a unit vector perpendicular to the plane of $\vec{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - \hat{k}$.

(f) Find the unit tangent vector at any point on the curve $x = t^2 + 1, y = 4t - 3, 2t^2 - 6t$.

(g) Define Ordinary point, Singular point and Regular singular point with example.

(h) Find the differential equation of all circles, which pass through the origin and whose center are on x-axis.

(i) Show that the given function $f(x,y) = xy^2$ satisfy Lipschitz condition on this domain $|x| \leq 1, |y| \leq 1$.

(j) Evaluate $\iint_S r.n\,dS$, where S is a closed surface.

(k) Find a first integral of $\frac{dy}{dx}\frac{d^2y}{dx^2} - x^2 y \frac{dy}{dx} = xy^2$.

(l) Solve : $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$.

(m) Find the radius of convergence of the power series

$$\sum \frac{(n+1)}{(n+2)(n+3)} x^n .$$

(n) Determine whether $x = 0$ is an ordinary point or a regular singular point of the differential equation

$$2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0.$$

(o) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, find the value of $\left| \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3} \right|$.

2. Answer any ***four*** questions from the following : 5×4=20

(a) Solve $(D^3 - D^2 + 3D + 5)y = x^2 + e^x \cos 2x$.

(b) Solve the differential equation

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x \,(x > 0)$ by the method of variation parameter.

(c) State the Green's theorem. Using this theorem evaluate the integral $\oint x^2 dx + (x + y^2) dy$ along the curve C : $y = 0$, $y = x$ and $y^2 = 2 - x$ in the first quadrant.

(d) Solve the system of linear equation :

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \text{ and } 2\frac{dx}{dt} + \frac{dy}{dt} + 3y - 2$$

(e) Find the solution to the initial value problem

$$y'' + y' - 12y = e^t + e^{2t} - 1; y(0) = 1, y'(0) = 3$$

P.T.O.

(f) It is given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are two linear independent solution of the associated homogeneous equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$. Find the particular integral and the general solution.

3. Answer any ***two*** questions from the following : 10×2=20

(a) (i) The acceleration of a particle at any time $t \geq 0$ is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\hat{i} - 8\sin 2t\hat{j} + 16t\hat{k}.$$

If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{v}$ and $\vec{r}$ at any time.

(ii) Evaluate $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2} dt$, given that $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$. (3+3)+4

(b) (i) If $\vec{F} = (xy - 3x^2)\hat{i} + (y + 2x)\hat{j} + 3xz^2\hat{k}$, evaluate the line integral $\int \vec{F}.d\vec{r}$ along the curve C : the straight lines from (1, 1, 1) to (2, 1, 1) then to (2, 2, 1) and then to (2, 2, 2).

(ii) Evaluate $\iiint xyz\, dxdy\, dz$ over the region $R : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$. 6+4

(c) (i) Solve : $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

(ii) Solve :

$$(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0.$$

6+4

(d) (i) Find the power series solution of the equation $(x^2 + 1)y'' + xy' - xy = 0$ in power of x (that is, about $x = 0$).

(ii) Solve : $(D^2 - 3D + 2)y = e^x$. 6+4

P.T.O.

OR

(Group Theory - I)

1. Answer any ***ten*** from the following questions : 2×10=20

(a) If $(G, \circ)$ be a group such that for a, b in G, $a \circ b = b \circ a^{-1}$ and $b \circ a = a \circ b^{-1}$. Show that $a^4 = b^4 = e$.

(b) Let H be a subgroup of G, then prove that $K = \{xax^{-1} : x \in G, a \in H\}$ is a subgroup of G.

(c) If $(G, \circ)$ be a finite group of even order, then show that G has an odd number of elements of order 2.

(d) If $\circ(H) = 6$ and $\circ(G) = 12$ where H is a subgroup of the group G, then show that H is normal in G.

(e) If $x = (1\ 2\ 3)$, $y = (2\ 4\ 3)$, $z = (1\ 3\ 4)$ then find $z\ y\ x$.

(f) Find all cyclic subgroups of the Klein's 4-group.

(g) Find all the elements of order 10 in the group $(Z_{30}, +)$.

(h) If N, M are normal in G and $\frac{G}{N} = \frac{G}{M}$, then show that $N = M$.

(i) Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi : G \to G'$ be a homomorphism. Show that $\phi(a^{-1}) = \{\phi(a)\}^{-1}, \forall a \in G$.

(j) If each element in a group be its own inverse, prove that the group is abelian.

(k) Let $(G, \circ)$ be a group and a, $b \in G$. If $\circ(a) = 3$ and $a \circ b \circ a^{-1} = b^2$, find $\circ(b)$ if $b \neq e$.

(l) If index of H in G be a prime number, then show that the quotient group G/H is cyclic.

(m) Show that the group and $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic.

(n) Let $(H, \circ)$ be a subgroup of $(G, \circ)$. Then show that the identity element of $(H, \circ)$ is the identity element of $(G, \circ)$.

(o) If $G = S_3$ and $H = A_3$ then find $[G : H]$.

2. Answer any ***four*** from the following questions : 5×4=20

(a) Show that the set $\mathbb{Q}^+$, set of all positive rationals forms an abelian group under '*' defined by $a * b = \frac{ab}{2}, \forall a, b \in \mathbb{Q}^+$.

P.T.O.

(b) Let G be a group and H_1, H_2 be two subgroups of G. Show that $H_1 \cap H_2$ is a subgroup of G, but $H_1 \cup H_2$ may not be a subgroup of G.

(c) Show that the order of a cyclic group is same as the order of its generator.

(d) Let G be a group in which $(ab)^3 = a^3b^3$, $\forall$ $a, b \in G$. Prove that $H = \{x^3 : x \in G\}$ is a normal subgroup of G.

(e) Find all homomorphisms from the group $(Z_6, +)$ to $(Z_4, +)$.

(f) Show that every group of prime order is cyclic.

3. Answer any ***two*** from the following questions : 10×2=20

(a) (i) Show that every permutation p on a finite set S is a cycle or it can be expressed as a product of disjoint cycles. 6

(ii) In a group $(G, \circ)$, show that $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$, $\forall\, a, b \in G$. 4

(b) (i) Let $(G, \circ)$ be a group. Prove that a non-empty subset H of G forms a subgroup of $(G, \circ)$ iff $a \in H,\ b \in H \Rightarrow a \circ b^{-1} \in H$. 6

(ii) Show that the centre $Z(G)$ of a group G is a normal subgroup of G. 4

(c) (i) State and prove Lagrange's theorem. 6

(ii) Show that any two left cosets of H in a group G have the same cardinality. 4

(d) (i) State and prove third law of isomorphism. 6

(ii) Let $\phi : (G, \circ) \rightarrow (G', *)$ be a homomorphism. Show that ker ϕ is a normal subgroup of $(G, \circ)$. 4

P.T.O.

OR

(Theory of Real Functions and Introduction to Metric Space)

1. Answer any ***ten*** questions : 2×10=20

(I) State Darboux's theorem.

(II) Write the geometrical interpretation of Rolle's theorem.

(III) Show that the function f defined by $f(x) = \sin x, x \in R$ is uniformly continuous on R.

(IV) Define interior point and adherent point of a set.

(V) Does $d(x, y) = (x - y)^2$ define a metric on the set of real numbers?

(VI) If d is a metric on X, then show that $\min\{d(x, y), 1\}$ is bounded metric on X.

(VII) Prove $\cos x > x - \frac{x^2}{2}$, if $0 < x < \frac{\pi}{2}$.

(VIII) State Taylor's theorem with Cauchy's form of remainder.

(IX) Define Pseudo-metric space.

(X) Define metric space.

(XI) Define uniform continuity.

(XII) Give an example of a function which is continuous on a closed interval but not bounded there.

(XIII) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(XIV) Show that $\lim_{x \to 0} x \cos \frac{1}{x} = 0$.

(XV) Show that sinx is continuous function for all real values of X.

2. Answer any ***four*** questions : 5×4=20

(I) Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ in $1 \le x \le 4$.

(II) Using Mean value theorem show that $\frac{x}{1+x} < \log(1+x), x > 0$.

(III) Find the expansion of the function $\log(1 + x)$ indicating the range.

(IV) In the Mean value theorem

$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1$, show that the limiting value of θ as $h \to 0^+$ is $\frac{1}{2}$ according as $f(x)$ is $\cos x$.

(V) Prove that the function $f(x) = x^3 + x^2 + x + 1$ has neither a maximum nor a minimum.

P.T.O.

(VI) Show that between any two real roots of $e^x \cos x = 1$, there exist at least one real root of $e^x \sin x - 1 = 0$.

3. Answer any *two* questions : 10×2=20

(I) State and prove Lagrange's Mean value theorem. Give its geometrical significance. 8+2

(II) State Cauchy's mean value theorem. Calculate ξ in Cauchy's mean value theorem for the functions

$f(x) = \sin x, g(x) = \cos x,$ on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. 2+8

(III) If (X, d) is a metric space then $\left(X, \frac{d}{1+d}\right)$ is also a metric space. Prove that arbitrary intersection of a closed set in a metric space is a closed set. 4+6

(IV) Define open sphere and closed sphere. Prove that, in a metric space, any open sphere is an open set. 2+8
