2022

3rd Semester Examination

MATHEMATICS (Honours)

Paper : C 6-T

(Group Theory - I)

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

1. Answer any ***ten*** questions : 2×10=20

(i) Define abelian group. Give example of a finite abelian group.

(ii) If G is a group of even order, then prove that it has an element $a \neq e$ such that $a^2 = e$.

(iii) Does the set of all odd integers form a group with respect to addition? Give suitable justification.

(iv) Suppose that a group contains elements a and b such that $O(a) = 4$, $O(b) = 2$ and $a^3b = ba$. Find $O(ab)$.

P.T.O.

(v) Find the order of $\alpha\beta$, if the permutations

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 2 & 7 & 6 & 1 & 5 \end{pmatrix} \text{ and}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 2 & 7 & 8 & 1 & 6 & 4 \end{pmatrix}$$

(vi) Show that the group $[\{1,2,3,4\}, X_5]$ is a cyclic group.

(vii) If a and x are two elements of a group G such that $axa^{-1} = b$, then find x. If $b^n = e$, then find x^n.

(viii) Let G and G' be two groups and $\theta : G \to G'$ be a homomorphism of G onto G'. Prove that if G is cyclic, then G' is also cyclic.

(ix) Let G be a group and H be a subgroup of G. Prove that $H = hH$ if and only if $h \in H$.

(x) Prove that every cyclic group is abelian, but converse is not true in general.

(xi) Define center of a group. If G be a group of order 4, what will be its center.

(xii) Define quotient group.

(xiii) Consider the group $G = GL(2, R)$ under multiplication and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Find centralizer of A, i.e., $C(A)$.

(xiv) Prove that intersection of two normal subgroups is normal.

(xv) Show that the direct product $Z_6 \times Z_4$ is not cyclic group.

2. Answer any ***four*** questions : 5×4=20

(i) Prove that the set of matrices

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$ where α is a real number,

forms a group under matrix multiplication. Does it abelian group?

(ii) Define even and odd permutation. Let α and β belong to S_n. Prove that $\beta\alpha\beta^{-1}$ and α both are either even or odd permutation together.

(iii) Define centralizer of an element in a group. Prove that for each a in a group G, the centralizer of a is a subgroup of G.

(iv) How many elements of order 9 does $Z_3 \oplus Z_9$ have?

(v) Find all the homomorphism of the group $(Z,+)$ to the group $(Z,+)$.

(vi) Prove that every group of prime order is cyclic.

3. Answer any ***two*** questions : 10×2=20

(i) (a) Let $n > 1$ be a fixed integer and let

P.T.O.

$u(n)=\{m \in N \mid m<n,(m,n)=1\}$. Then prove that $(u(n), X_n)$ is a group. If $n = 100$, then what is the order of $u(n)$?

(b) Define Alternating group of order n. Find all elements of A_3. 5+5

(ii) (a) State and prove Lagrange's theorem on groups. By using Lagrange's theorem prove that if H and K are subgroups whose orders are relatively prime, then show that $H \cap K=\{e\}$.

(b) How many generators are there of the cyclic group of order 8? (2+4+2)+2

(iii) (a) Let G be a finite abelian group and let p be a prime number such that p divides order of G Then prove that G has an element of order p. (Cauchy's Theorem).

(b) Prove that any group of order four is abelian. 5+5

(iv) (a) Let $\theta: G \rightarrow G'$ be a homomorphism of a group G onto a group G'. Let $K=\ker\theta$. Then prove that K is a normal subgroup of G and $\frac{G}{K} \cong G'$. (First Isomorphism Theorem).

(b) Let H be a subgroup of G. If $x^2 \in H$ for all $x \in G$, then prove that H is a normal subgroup of G and G/H is commutative. 5+5