

2022



3rd Semester Examination
MATHEMATICS (Honours)

Paper : C 5-T

(Theory of Real Functions and
Introduction to Metric Space)

[CBCS]

Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

1. Answer any *ten* questions : 2×10=20(a) Prove that, $\frac{\tan x}{x} > \frac{x}{\sin x}$, $0 < x < \frac{\pi}{2}$

(b) Use Mean value theorem to show that

$$0 < \frac{1}{x} \log \left(\frac{e^x - 1}{x} \right) < 1$$

(c) Give the geometrical interpretation of Mean Value Theorem.

(d) Does there exist a function ϕ such that $\phi'(x) = f(x)$ on $[0, 2]$, when $f(x) = x - [x]$, where $[x]$ is the greatest integer function?

P.T.O.

(2)

- (e) Let $I \in \mathbb{R}$ be an interval and a function $f: I \rightarrow \mathbb{R}$ be differentiable at $c \in I$. Then if $f'(c) > 0$, prove that f is increasing at c .

(f) Evaluate $\lim_{x \rightarrow 3} \left([x] - \left[\frac{x}{3} \right] \right)$

(g) Let $f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in \mathbb{R} - Q \end{cases}$

Prove that f is discontinuous at every point c in \mathbb{R} .

(h) Define the Lipschitz's function.

(i) Find the value of limit : $\lim_{x \rightarrow \alpha} \left(1 + \frac{3}{x} \right)^x$

(j) Let d_1 and d_2 are two metrics on a non-empty set A . Prove that $d_1 + d_2$ is also a metric on A .

(k) Let (M, d) be a metric space. Then prove that $\forall A, B \in M, A \subset B \Rightarrow \delta(A) \geq \delta(B)$.

(l) Let (A, d) be a metric space. Then prove that (A, \sqrt{d}) is also a metric space.

(m) Prove that in any discrete metric space all the sets are closed.

(n) Let X be a non-empty set and $f: X \rightarrow \mathbb{R}$ be an injective function. Then prove that $d(x, y) = |f(x) - f(y)| \forall x, y \in X$ defines a metric on X .

(3)

(o) Show that $\lim_{x \rightarrow \infty} \frac{[x]}{x} = 1$, where $[x]$ denotes the greatest integer contained in x not greater than x .

Group - B

2. Answer any *four* questions :

5×4=20

(a) Let $[a, b]$ be a bounded closed Interval and I denoted the set of all Riemann-Integrable (\mathbb{R} I) function over $[a, b]$. Then prove that

$$d(f, g) = \int_a^b |f(x) - g(x)| dx, \forall f, g \in \mathbb{R}I$$

is a pseudo metric but not metric.

(b) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$ but $f(x) = \sin x$ is uniformly continuous on $[0, \infty)$.

5

(c) State & prove the Hausdorff property.

(d) Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Then $\lim_{x \rightarrow c} f(x) = l$ iff for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c , the sequence $\{f(x_n)\}$ converges to l .

(e) State and prove the Caratheodory's theorem.

(4)

(i) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable on \mathbb{R} but f' is not continuous on \mathbb{R} .

Group - C

3. Answer any two questions :

10×2=20

- (a) (i) State Maclaurin's infinite series and obtain the expansion of $(1+x)^m$ where m is any real number other than positive integer and $|x| < 1$.
5+5
- (ii) Let $f: I \rightarrow \mathbb{R}$ be such that f has a local extremum at an interior point C of I . If $f'(c)$ exists then $f'(c) = 0$.
5+5
- (b) (i) Let I be an interval and a function $f: I \rightarrow \mathbb{R}$ be uniformly continuous on I . Then f is continuous on I .
- (ii) Let $I = [a, b]$ be a closed and bounded interval and a function $f: I \rightarrow \mathbb{R}$ be continuous on I . Then f is uniformly continuous and f is bounded on I .
2+8
- (c) (i) Let $I = (a, b)$ be a bounded open interval and $c \in (a, b)$. If $f: I \rightarrow \mathbb{R}$ be a monotonic function on I then $\lim_{x \rightarrow c^-} f(x)$ and $\lim_{x \rightarrow c^+} f(x)$ both exist.

(5)

(ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in \mathbb{R} .
5+5

(d) (i) State and prove Taylor's theorem with Cauchy's form of remainder after n terms.

(ii) Give examples of a function which is :

- (a) Continuous and bounded on \mathbb{R} , attains its supremum but not infimum.
- (b) Continuous and bounded on \mathbb{R} , attains its infimum but not its supremum.