বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 10 - T

Ring Theory and Linear Algebra - I

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

1. Answer any ***five*** questions : 2×5=10

(a) Let X be any set and R be the power set of X. Does $(R, +, \cdot)$ form a ring where $A+B=A\cup B$ and $A\cdot B=A\cap B$ for all $A,B\in R$.

(b) Extend $S = \{(1,1,0), (1,1,1)\}$ to a basis of the vector space $\mathbb{R}^3$ over $\mathbb{R}$.

(c) Find the total number of units in the ring $M_2(Z_3)$, with usual notations.

(d) Let V be a vector space over the field F and W_1, W_2 be two subspaces of V. Is $W_1 \cup W_2$ a subspace of V?

(e) In the ring $M_2(\mathbb{Z})$ of all 2 × 2 matrices over $\mathbb{Z}$, check whether the set $\left\{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a,b\in\mathbb{Z}\right\}$ forms an ideal or not.

(f) Determine the linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ over $\mathbb{R}$ such that T sends the vectors (1,0,0), (0,1,0), (0,0,1) to (0,1,0), (0,0,1), (1,0,0), respectively.

(g) Show that the mapping $f:\mathbb{Z}\sqrt{2} \to M_2(\mathbb{R})$ (where $M_2(\mathbb{R})$ denotes the ring of all 2 × 2 real matrices) defined by $f\left(a+b\sqrt{2}\right)=\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ is a homomorphism of rings.

(h) Give an example of a linear operator T on a vector space V such that ker T = *Im T*.

2. Answer any ***four*** questions : 5×4=20

(a) Let I denote the set of all polynomials in $\mathbb{Z}[x]$ with constant term of the form $4k\ (k \in \mathbb{Z})$. Show that I is an ideal of $\mathbb{Z}[x]$. Is it a prime ideal? Is it a maximal ideal? Give proper justification in support of your answer. 2+2+1=5

(b) Let V be a vector space over the field F and $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ be a basis for V. Let $\beta \in V$ be a non-null vector such that $\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots. + c_n\alpha_n$ for some $c_1, c_2, \ldots\ldots, c_n \in F$ where $c_k \neq 0$ for some $1 \leq k \leq n$. Then show that $\{\alpha_1, \alpha_2, \ldots.., \alpha_{k-1}, \beta, \alpha_{k+1}, \ldots., \alpha_n\}$ is also a basis for V. 5

(c) Let (F, + , .) be a field and $u(\neq 0) \in F$. Define multiplication × in F by $a \times b = a.u.b$ for $a, b \in F$. Prove that (F, +, ×) is a field.

(d) Find a non-identity linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ such that the $T(W) = W$ where $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$.

(e) Let R be a ring and S be a non-empty subset of R. Show that $M = \{a \in R \mid ax = 0 \text{ for all } x \in S\}$ is a left ideal of R. Give an example to show that M need not be always an ideal of R. 2+3=5

(f) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(a,b,c) = (a+b, 2c-a)$ for all $(a,b,c) \in \mathbb{R}^3$. Find the matrix representation of T relative to the pair of bases $B = \{(1,0,-1),(1,1,1),(1,0,0)\}$ and $B' = \{(0,1),(1,0)\}$.

3. Answer any ***three*** questions : 10×3=30

(a) (i) Let R be the ring of all continuous function from $\mathbb{R}$ to $\mathbb{R}$. Show that $A=\{f \in R \mid f(0)=0\}$ is a maximal ideal of R.

(ii) Check whether the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$ is isomorphic.

(iii) Let V be a real vector space with three subspaces P, Q, R satisfying $V=P\cup Q\cup R$. Prove that at least one of P, Q, R must be V itself.

3+3+4=10

(b) (i) Let I and J be two ideals of a ring R. Find the smallest ideal of R containing both I and J.

(ii) Give an example to show that quotient ring of an integral domain is not always an integral domain.

(iii) The matrix of a linear operator $T:\mathbb{R}^3\to\mathbb{R}^3$ with respect to the ordered basis $B=\{(-1,1,1),(1,-1,1),(1,1,-1)\}$ is $A=\begin{pmatrix}1&2&2\\2&1&3\\3&3&1\end{pmatrix}$. Find the matrix of T with respect to the ordered basis $B_1=\{(0,1,1),(1,0,1),(1,1,0)\}$.

3+2+5=10

(c) (i) Let $S=\{\frac{a}{b}\in\mathbb{Q} \mid \gcd(a,b)=1$ and 3 does not divide $b\}$. Show that S is a ring under usual addition and multiplication of rational numbers. Also prove that $M=\{\frac{a}{b}\in S \mid 3$ divides $a\}$ is an ideal of S and the quotient ring S/M is a field.

(ii) Let V, W be two finite dimensional vector spaces over the same field $F, T:V\to W$ be a linear transformation. Then prove that following are equivalent : (A) T carries each linearly independent subset of V to a linearly independent subset of W. (B) $\ker T=\{\theta\}$. (2+2+2)+4=10

(d) (i) Prove that the ring Z_n is s principal ideal ring.

(ii) Find all non-trivial ring homomorphisms from the ring $\mathbb{Z}_{12}$ to the ring $\mathbb{Z}_{28}$.

(iii) Let U, V, W be three finite dimensional vector spaces over the field $F, T : V \to W$ be a linear transformation and $S : W \to U$ be an isomorphism. Then prove that (A) dim ker T = dim ker ST and (B) dim ImT = dim Im ST. 3+3+4=10

(e) (i) In a commutative ring R with unity, then show that an ideal P is a prime ideal if and only if the quotient ring $\frac{R}{P}$ is an integral domain.

(ii) Give an example of an infinite ring with finite characteristic.

(iii) Let U and W be two subspaces of a vector space V over the field F. Prove that $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.

4+2+4=10