Total pages:2

PG CBCS

M.SC. Semester-IV Examination, 2021

MATHEMATICS

PAPER: MTM-401

(**FUNCTIONAL ANALYSIS**)

Full Marks: 40 **Time: 2 Hours**

Answer any FOUR questions from the following: **4×10=40**

1. (a) Let X≠{0} be a normed linear space. Prove that X is a Banach space if and only if {x: ‖x‖=1} is complete.

   (b) Prove that a normed linear space X is finite dimensional if and only if the closed unit sphere in X is complete. **5+5**

2. Let H is a Hilbert space and H^* be its dual space.If fϵ H^* is an arbitrary butfixedfunctional, then there exists a unique vector zϵH such that f(x)=<x,z> for all x in H , Where z depends on f and has the norm ‖z‖=‖f‖. **10**

3. (a) State Open Mapping Theorem. Show that $T : \mathbb{R}^2 \to \mathbb{R}$ be defined by $T(x, y) = x$for $(x, y) \in \mathbb{R}^2$ is an open mapping. Is T^{-1} bounded, if exists?

   (b)Suppose $X = C^1[0,1]$, i.e. the set of all functions $f: [0,1] \to \mathbb{C}$ such that f' exists and is continuous. Let $Y = C[0,1]$ and let X and Y be equipped with supremum norm. Define $A: X \to Y$ by $Af = f'$. Show that the graph of A is closed. **5+5**

4. (a) Let X and Y be inner product spaces. Then a linear map $F: X \to Y$ satisfies $\langle F(x), F(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$ if and only if it satisfies $\|F(x)\| = \|x\|$ for all $x \in X$, where the norms on X and Y are induced by the respective inner products.

   (b) Let $P \in BL(\mathcal{H})$ be a nonzero projection on a Hilbert space $\mathcal{H}$and $\|P\| = 1$. Then show that P is an orthogonal projection. **5+5**

5. (a)Let X and Y be Banach spaces and $A \in BL(X,Y)$. Show that there is a constant $c > 0$ such that $\| Ax \| \geq c \| x \|$ for all $x \in X$ if and only if $Ker(A) = \{0\}$ and $Ran(A)$ is closed in X.

   (b) Let $T : C[0,1] \to C[0,1]$ be defined by $T(t) = \int_0^t x(\tau)d\tau$. Find $R(T)$ and obtain $T^{-1}: R(T) \to C[0,1]$. Examine whether T^{-1} is linear and bounded. **5+5**

6. (a) Let the space $l^2(\mathbb{Z})$ be defined as the space of all two- sided square summable sequences and the bilateral shift is the operator W on $l^2(\mathbb{Z})$ defined by $W(\dots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \dots) = (\dots, a_{-3}, a_{-2}, \hat{a}_{-1}, a_0, a_1, \dots)$. Prove that

   (i) W is unitary, and

   (ii) the adjoint W^* of W is given by $W^*(\dots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \dots) = (\dots, a_{-1}, a_0, \hat{a}_1, a_2, a_3, \dots)$.

   (b) Show that $< Ae_j, e_i >= (i + j + 1)^{-1}$ for $0 \leq i, j \leq \infty$ defines a bounded operator on $l^2(\mathbb{N} \cup \{0\})$ with $\| A \| \leq \pi$. **7+3**

7. (a) When is a series in a normed linear space said to be convergent and absolutely convergent? Prove that a normed linear space X is a Banach space if and only if every absolutely convergent series in X is convergent.

   (b) If $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series in a Banach space X then show that $\| \sum_{n=1}^{\infty} x_n \| \leq \sum_{n=1}^{\infty} \| x_n \|$ **7+3**

8. (a) If M is a closed linear subspace of a normed linear space N and u is a vector not in M , then there exists a functional g in N^* such that g(M)=0 and g(u)≠0.

   (b) If f is a non-zero linear functional on an infinite dimensional linear space X, does there exist a norm on X such that f is discontinuous?

   5+5

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