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PG CBCS
M.SC. Semester-I Examination, 2021
MATHEMATICS
PAPER: MTM 101

(REAL ANALYSIS)

Full Marks: 50 **Time: 2 Hours**

Answer any <u>FOUR</u> questions of the following: **10X4=40**

1. a) Show that the conditions
   i) $d(x,y) = 0$ iff $x = y$ $(x, y \in X)$ and
   ii) $d(x,y) \leq d(x,y) + d(y,z), \forall x, y, z \in X$
   are not sufficient to ensure that the map $d: X \times X \to \mathbb{R}$ is a metric on the set X.
   b) Show that the function $d: C[0,1] \times C[0,1] \to \mathbb{R}$ defined by $d(f,g) = \inf\{|f(t) - g(t)|: t \in [0,1]\}$ where $f, g \in C[0,1]$(=the set of all real valued continuous function over $[0,1]$) is not a metric on $C[0,1]$.
   c) i) Define an equivalent metric.
   ii) Prove that two metrics d_1 and d_2 on a non-empty set X are equivalent if there exist real numbers $r, s > 0$ such that $d_1(x,y) \leq r\, d_2(x,y)$ and $d_2(x,y) \leq s\, d_1(x,y)$ $\forall$ $x, y \in X$. $3 + 2 + (2 + 3)$

2. a) i) Define a finite intersection property.
   ii) Prove that a metric space (X, d) is compact if and only if for every collection of closed set $\{F_\alpha : \alpha \in \Lambda\}$ in X possessing finite intersection property, the intersection $\bigcap_{\alpha \in \Lambda} F_\alpha$ of the entire collection is nonempty.
   b) i) Prove that any continuous function on a compact metric space (X, d_1) to a metric space (Y, d_2) is uniformly continuous.
   ii) Determine whether the set $S = \{(x,y): 0 < x \leq 1, x^2 + y^2 = 4\}$ is compact in $\mathbb{R}^2$. $(2 + 3) + (3 + 2)$

3. a). Prove that (X, d) is connected iff the only subsets of X, both open and closed, are X and ϕ.
   b). Show that every continuous function $f: [-1,1] \to [-1,1]$ of the closed interval $[-1,1]$ into itself has at least one fixed point, i.e., $\exists$ $x \in [-1,1]$ such

that $f(x) = x$.

c). Examine whether the following subsets of $\mathbb{R}^2$ (with usual metric) are connected.

i) $A = \{(x,y): x > 0,\ x^2 + y^2 \leq 1\} \cup \{(x,y): x < 0,\ x^2 + y^2 \leq 1\}$.

ii) $B = \{(x,y): x = 0, 0 < y < 1\} \cup \{(x,y): 0 < x < 1, y = 0\}$.

$3 + 3 + (2 + 2)$

4. Examine if the following statements are true or false with proper justification.

$2 \times 5 = 10$

i) Every Cauchy sequence is convergent in $(\mathbb{R}, d)$, where $d(x,y) = |x - y|$ for $x, y \in \mathbb{R}$.

ii) Let X denote the set of all Riemann integrable functions on $[a,b]$. (X,d) is a metric space where $d(f,g) = \int_a^b |f(x) - g(x)| dx$ for $f, g \in X$.

iii) A is a compact set in a metric space (X,d) and $b \in X \setminus A$. Then does not exists any point $a \in A$ such that $d(a,b) = d(A,b)$.

iv) Let A and B be two closed sets in a metric space (X, d_1) such that $X = A \cup B$, and let $f: A \to Y$ and $g: B \to Y$, where (Y, d_2) is a metric space, be continuous maps such that $f(x) = g(x), \forall x \in A \cap B$. Then the map $h: X \to Y$, defined by $h(x) = \begin{cases} f(x)\ if\ x \in A \\ g(x)\ if\ x \in B \end{cases}$ is continuous.

v) For any subset $A \subseteq X, \chi_A \in \mathbb{L}_0^+$. Where $\chi_A(x) = \begin{cases} 1\ if\ x \in A \\ 0\ if\ x \notin A \end{cases}$ and $\mathbb{L}_0^+$ is the class of all non-negative simple measurable functions on X.

5. a). Establish a necessary and sufficient condition for a function $f: [a,b] \to \mathbb{R}$ to be a function of bounded variation on $[a,b]$.

b). Show that the set of all functions of bounded variation on $[a,b]$ forms a vector space under usual addition and multiplication by scalars. 5+5

6. a). If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then show that $\alpha \in \mathcal{R}(f)$ on $[a,b]$. Also show that $\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a)$.

(b). Show that every finite sum of real numbers can be expressed as the R-S integral over some interval. 5+5

7. a). Suppose $f: X \to [0, \infty]$ is measurable and $\phi(E) = \int_E f\, d\mu$ for every measurable set E in X. Show that ϕ is a measure and $\int g\, d\phi = \int gf\, d\mu$ for every measurable function g on X with range in $[0, \infty]$.

b). If $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in X$, then show that $\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$. 5+5

8. (a). Let $f(x) = \frac{1}{x^p}$ if $0 < x \leq 1$ and $f(0) = 0$. Find necessary and sufficient condition on p such that $f \in L^1[0, 1]$. Compute $\int_0^1 f(x)\lambda(x)$ in that case.

(b). Evaluate the following: $\int_{-1}^{3} 2 \ \cos x \, d(2x + [x])$. 7+3

[**Internal Assessment-10 Marks**]