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UG/3rd Sem/MATH(H)/19

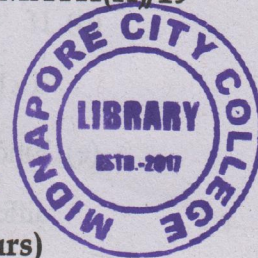
2019

B.Sc.

3rd Semester Examination

MATHEMATICS (Honours)

Paper - GE 3-T



Full Marks : 60

Time : 3 Hours

*The figures in the margin indicate full marks.  
Candidates are required to give their answers  
in their own words as far as practicable.  
Illustrate the answers wherever necessary.*

**Differenrial Equation and Vector Calculus**

1. Answer any ten questions : 10×2

(a) Determine whether the set  $\{1 - x, 1 + x, 1 - 3x\}$  is linearly dependent on  $(-\infty, \infty)$ .

(b) Find  $\frac{1}{D^2 + 4}(\sin 2x)$ , where  $D^2 = \frac{d^2}{dx^2}$ .

(c) Find the vector area of the triangle, the position vectors of whose vertices are  $\hat{i} + \hat{j} + 2\hat{k}$ ,  $2\hat{i} + 2\hat{j} + 3\hat{k}$  and  $3\hat{i} - \hat{j} - \hat{k}$ . [ Turn Over ]

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(d) If  $\vec{a} + \vec{b} + \vec{c} = 0$ , then prove that  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ .

(e) Find the equilibrium point of the system of differential equations  $\dot{x} = e^{x-1} - 1$  and  $\dot{y} = ye^x$ .

(f) State the principle of superposition for homogeneous equation.

(g) If  $u$  and  $v$  be two independent solutions of the linear equation  $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ , then the Wronskian  $w(u, v)$  is given by  $w(u, v) = Ae^{-\int P dx}$ , where  $A$  is a constant.

(h) Show that the three vectors  $4\hat{i} + 2\hat{j} + \hat{k}$ ,  $2\hat{i} - \hat{j} + 3\hat{k}$  and  $8\hat{i} + 7\hat{k}$  are coplanar.

(i) Find the directional derivative of  $\phi = x^2yz + 4xz^2$  at  $(1, -2, -1)$  in the direction  $2\hat{i} - \hat{j} - 2\hat{k}$ .

(j) Verify that  $x = 0$  is a singular point of the differential equation

$$2x^2 \frac{d^2w}{dx^2} + x \frac{dw}{dx} - (x+1)w = 0.$$

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(k) Find constants  $a, b$  and  $c$  so that

$$\vec{F} = (x + 2y + az)\hat{i} - (bx - 3y - z)$$

$$\hat{j} + (4x + cy + 2z)\hat{k} \text{ is irrotational.}$$

(l) A particle moves along the curve  $x = 2t^2$ ,  $y = t^2 - 4t$ ,  $z = 3t - 5$ , where  $t$  is the time. Find the components of its velocity and acceleration at time  $t = 1$  in the direction  $\hat{i} - 3\hat{j} + 2\hat{k}$ .

(m) State Lipschitz condition.

(n) Define Euler-cauchy type of equation.

(o) If  $\frac{d\vec{a}}{dt} = \vec{r} \times \vec{a}$  and  $\frac{d\vec{b}}{dt} = \vec{r} \times \vec{b}$ , then show that

$$\frac{d}{dt}[\vec{a} \times \vec{b}] = \vec{r} \times \vec{a} \times \vec{b}$$

where  $\vec{r}$  is a constant vector and  $\vec{a}$  and  $\vec{b}$  are the vector function of a scalar variable  $t$ .

2. Answer any four questions : 4×5  
(a)

Solve the differential equation  $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$  by the method of variation parameter.

[ Turn Over ]

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(b) Solve :  $\frac{dx}{y-zx} = \frac{dy}{x+yz} = \frac{dz}{x^2+y^2}$

(c) Solve :  $\frac{dx}{dt} - 7x + y = 0$ ;  $\frac{dy}{dt} - 2x - 5y = 0$

(d) Show that the volume of the parallelopiped, whose edges are represented by  $3\hat{i} + 2\hat{j} - 4\hat{k}$ ,  $3\hat{i} + \hat{j} + 3\hat{k}$  and  $\hat{i} - 2\hat{j} + \hat{k}$  is 49 cubic units.

(e) (i) Evaluate  $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$  along the path  $x^4 - 6xy^3 = 4y^2$ .

(ii) Show that  $\nabla r^n = nr^{n-2}\vec{r}$ .

(f) (i) If W be the wronskian of the functions 1, x, x<sup>2</sup>, ..., x<sup>n-1</sup> for n > 1, then show that W = 0! 1! 2! ..... (n - 1)!

(ii) Obtain the differential equation of all circles, each of which touches the axis of x at the origin.

3. Answer any two questions : 2×10

(a) Find the series solution of the equation  $(x^2 + 1)y_2 + xy_1 - xy = 0$  about  $x = 0$ .

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(b) (i) If  $\vec{r} = (a \cos t)\hat{i} + (a \sin t)\hat{j} + (at \tan \alpha)\hat{k}$ , then

prove that  $\left[ \frac{d\vec{r}}{dt} \frac{d^2\vec{r}}{dt^2} \frac{d^3\vec{r}}{dt^3} \right] = a^3 \tan \alpha$ . 5

(ii) Prove that  $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2} dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$ ,

where  $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ . 5

(c) (i) If  $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ , evaluate  $\iint_S \vec{F} \cdot \hat{n} ds$

where S is the surface of the cube bounded by  $x = 0$ ,  $x = 1$ ,  $y = 0$ ,  $y = 1$ ,  $z = 0$ ,  $z = 1$ . 5

(ii) Given the space curve  $x = t$ ,  $y = t^2$ ,  $z = \frac{2}{3}t^3$ , find the curvature k and the torsion Y. 5

(d) Show that the model represented by

$$\frac{dx}{dt} = x(4-x-y); \quad \frac{dy}{dt} = y(15-5x-3y),$$

$x \geq 0$ ,  $y \geq 0$  has a position of equilibrium and this position is stable.

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Group Theory - I

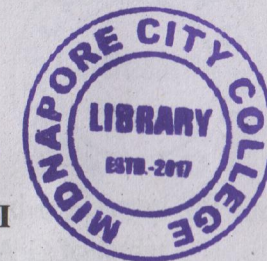
Unit - I

1. Answer any *two* questions :  $2 \times 2 = 4$

- (a) Prove that  $(\mathbb{Z}, +)$  is the semigroup.
- (b) If each element, except the identity, of a group be of order 2, prove that the group is abelian.
- (c) Define symmetric group  $S_3$ . What is the identity element of this group.

2. Answer any *one* question :  $5 \times 1 = 5$

- (a) In a group  $(G, \circ)$  in which  $(a \circ b)^3 = a^3 \circ b^3$  and  $(a \circ b)^5 = a^5 \circ b^5$  for all  $a, b \in G$ , prove that the group is abelian.
- (b) Prove that the set  $D$  of all odd integers forms a commutative group with respect to  $*$  defined by  $a * b = a + b - 1$  for  $a, b \in D$ .



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Unit - II

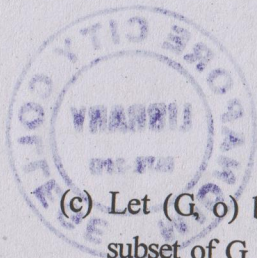
3. Answer any *two* questions :  $2 \times 2 = 4$

- (a) Find the Cyclic subgroups of the group  $(S, \circ)$ , where  $S = \{1, i, -1, -i\}$ .
- (b) Give an example to show that the union of two subgroups of a group may not be a sub group.
- (c) Let  $(G, \circ)$  be a group and  $K \subseteq G$ . If  $(H, \circ)$  be a subgroup of  $(G, \circ)$  and  $(K, \circ)$  be a subgroup of  $(H, \circ)$ , prove that  $(K, \circ)$  is a subgroup of  $(G, \circ)$ .

4. Answer any *two* questions :  $2 \times 5 = 10$

- (a) If  $H$  and  $K$  happens to be a subgroup of  $G$ , then prove that  $\circ(HK) = \frac{\circ(H) \cdot \circ(K)}{\circ(H \cap K)}$ .
- (b) Let  $G$  be a group in which  $(ab)^3 = a^3 b^3$  for all  $a, b \in G$ . Show that  $H = \{x^6 : x \in G\}$  is a subgroup of  $G$ .

[ Turn Over ]



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- (c) Let  $(G, \circ)$  be a group and  $H$  be a non-empty subset of  $G$ . A relation  $\rho$  defined on  $G$  by " $a\rho b$  if and only if  $a \circ b^{-1} \in H$  for  $a, b \in G$ ", is an equivalence relation on  $G$ . Prove that  $(H, \circ)$  is a subgroup of  $(G, \circ)$ .

### Unit - III

5. Answer any *two* questions : 2×2=4

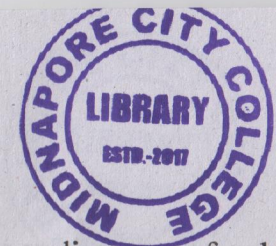
- (a) Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Let  $a \in G - H$ . The prove that  $aH \cap H = \phi$ .
- (b)  $G$  is a cyclic group of order 30 generated by  $a$ . Find the order of the cyclic subgroup generated by  $a^{18}$ .

(c) Find the inverse of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$$

6. Answer any *one* question : 1×10=10

- (a) (i) Prove that every group of prime order is cyclic.
- (ii) A cyclic group of finite order  $n$  has one and only one subgroup of order  $d$  for every positive divisor  $d$  of  $n$ . 5



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- (b) (i) Let  $G = \langle a \rangle$  be a cyclic group of order  $n$ . Prove that every subgroup  $H$  of  $G$  is of the form  $\langle a^m \rangle$  where  $m$  is a divisor of  $n$ . 5
- (ii) State and prove Fermat's Little theorem. 5

### Unit - IV

7. Answer any *two* questions : 2×2=4

- (a) Let  $G_1, G_2$  be two groups and  $Z(G_1), Z(G_2)$  be their respective centres. Prove that  $Z(G_1) \times Z(G_2)$  is a centre of the group  $G_1 \times G_2$ .

(b) Find the order of the element  $\frac{2}{3} + Z$  in the quotient group  $Q/Z$ .

(c) Let  $H$  be a subgroup of a group  $G$  and  $[G : H] = 2$ . Show that  $H$  is a normal subgroup of  $G$ .

8. Answer any *one* question : 1×10=10

- (a) (i) Show that subgroup  $H$  of a group  $G$  is normal if  $aHa^{-1} = H$  for every  $a$  in  $G$ . 5

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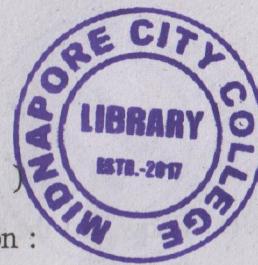
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- (ii) Let  $H$  be a subgroup of a group  $G$  and  $[G : H] = 2$ . Then show that  $H$  is normal in  $G$  5
- (b) Let  $a, b \in \mathbb{R}$  and a mapping  $T_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $T_{a,b}(x) = ax + b, x \in \mathbb{R}$ . Let  $G = \{T_{a,b} : a \neq 0\}$ . Prove that  $(G, \circ)$  is a group where  $\circ$  is the composition of mappings. Let  $H = \{T_{a,b} : a = 1\}$ . Prove that  $H$  is a normal subgroup of  $G$ .

### Unit - V

9. Answer any two questions : 2×2=4

- (a) Let  $G = (\mathbb{Z}, +)$  and a mapping  $\phi : G \rightarrow G$  be defined by  $\phi(x) = x + 1, x \in G$ . Examine if  $\phi$  is a homomorphism.
- (b) Show that the groups  $(\mathbb{Q}, +)$  and  $(\mathbb{R}, +)$  are not isomorphic.
- (c) Let  $(G, \circ)$  be a group and a mapping  $\phi : G \rightarrow G$  is defined by  $\phi(x) = x^2, x \in G$ . Prove that  $\phi$  is homomorphism if  $G$  is commutative.



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10. Answer any one question : 1×5=5

- (a) Let  $G = S_3, G' = (\{1, -1\}, \cdot)$  and let  $\phi : G \rightarrow G'$  be defined by
- $\phi(\alpha) = 1$  if  $\alpha$  be an even permutation in  $S_3$   
 $= -1$  if  $\alpha$  be an odd permutation in  $S_3$ .
- Examine if  $\phi$  is a homomorphism.

- (b) Let  $H \subset K \subset G$  and  $H$  is normal in  $K, K$  is normal in  $G$  and also  $H$  is normal in  $G$ . Show that  $K/H$  is normal in  $G/H$  and  $\frac{G/H}{K/H} = G/K$ .

### Theory of Real Function and Introduction

### Unit - I

1. Answer any three questions : 3×2=6

- (a) Given an example of jump discontinuity of a function.
- (b) Find the points of discontinuity of  $f(x) = \frac{1}{1 + \frac{1}{x}}$ .

[ Turn Over ]



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(c) Show that the absolute value function  $f(x) = |x|$  is continuous at every point  $c \in \mathbb{R}$ .

(d) If  $\lim_{x \rightarrow a} |f(x)| = 0$  then show that  $\lim_{x \rightarrow a} f(x) = 0$ .

(e) State intermediate value theorem.

2. Answer any one question : 1×5=5

(a) If  $f(x)$  and  $g(x)$  are two real valued functions of  $x$  defined on an interval  $I$  such that  $\lim_{x \rightarrow a} f(x) = l$

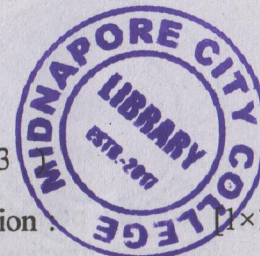
and  $\lim_{x \rightarrow a} g(x) = m$ , then prove that

$$\lim_{x \rightarrow a} \{f(x).g(x)\} = lm, a \in I.$$

(b) Discuss the continuity of the function

$$f(x) = \begin{cases} 2, & x^2 > 4 \\ 3, & x^2 = 4 \\ 0, & x^2 < 4 \end{cases}$$

State the type of discontinuity if  $f(x)$  is discontinuous any-where.



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3. Answer any one question . [1×10 = 10]

(a) (i) If  $f(x) = x$  and  $g(x) = \sin x$ , show that both  $f$  and  $g$  are uniformly continuous on  $\mathbb{R}$ , but that their product  $fg$  is not uniformly continuous on  $\mathbb{R}$ . 5

(ii) Show that

$$\lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right) = 1 \quad 5$$

(b) (i) Prove that if  $f$  be continuous on a closed interval, then it assumes its least upper bound and its greatest lower bound in that interval. 6

(ii) Prove that if a function  $f$  is uniformly continuous in a certain interval  $I$ , then it is necessarily continuous on  $I$ . 4

**Unit - II [Marks : 14]**

4. Answer any two questions : 2×2=4

(a) Show that  $x > \sin x$  for  $0 < x < \frac{\pi}{2}$ .

(b) What is the geometrical interpretation of the Rolle's theorem.

[ Turn Over ]



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- (c) Find the point of relative extrema of the following function on the specified domain :

$$f(x) = 1 - (x - 1)^{2/3} \text{ for } 0 \leq x \leq 2.$$

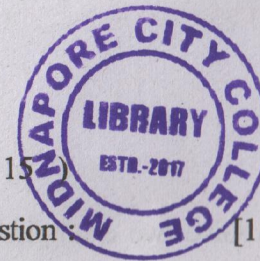
5. Answer any two questions : [2×5 = 10]

- (a) State and prove Rolle's theorem.
- (b) Verify Lagrange's Mean value theorem for the function  $f(x) = x(x - 1)(x - 3)$  in  $[0, 4]$ .
- (c) In the mean value theorem  $f(a + h) = f(a) + h \cdot f'(a + \theta h)$  if  $a = 1$ ,  $h = 3$  and  $f(x) = \sqrt{x}$ , find  $\theta$ .

**Unit - III [Marks : 14]**

6. Answer any two questions : 2×2=4

- (a) Show that the maximum value of  $(x)^x$  is  $(e)^{\frac{1}{e}}$ .
- (b) Examine the function  $f(x) = 4 - 3(x - 2)^{2/3}$  for maxima and minima at  $x = 2$ .
- (c) State Maclaurin's theorem with Cauchy's form of remainder.



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7. Answer any one question : [1×10 = 10]

- (a) (i) Find the minimum and maximum value of

$$f(x) = 3x + \frac{2}{3x} \text{ for all } x \in \mathbb{R} - \{0\}.$$

- (ii) Assuming the validity of expansion, show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \dots$$

- (b) (i) Show that  $\frac{\tan x}{x} > \frac{x}{\sin x}$  for  $0 < x < \frac{\pi}{2}$ .

- (ii) Show that  $R_n \rightarrow 0$  as  $n \rightarrow \infty$  for the expansions of  $(1 + x)^m$  in a given range of validity, where  $R_n$  is the remainder after  $n$  terms.

**Unit - IV [Marks : 11]**

8. Answer any three questions : 3×2=6

- (a) Define dense set.
- (b) Let  $(X, d)$  be a metric space. Then show that diameter  $\delta(A) = 0$  iff  $A \subset X$  is a singleton set.
- (c) Define open ball. Give an example.

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(d) Give two examples of separable metric spaces.

(e) Let  $(X, d)$  be a metric space. Then prove that

(i) the null set  $\phi$  is closed,

(ii)  $X$  is closed.

9. Answer any one question : [1×5 = 5]

(a) Prove that every non-empty open set in the real line is the union of a countable collection of mutually disjoint open intervals.

(b) If  $d$  is a metric on a set  $X$ , then proved that  $d_1$  and  $d_2$  are metric where

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

$$d_2(x, y) = \min\{1, d(x, y)\} \text{ for } x, y \in X.$$