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2019

B.Sc.

3rd Semister Examination

MATHEMATICS (Honours)

Paper - GE 3-T

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Differenrial Equation and Vector Calculus

1. Answer any ten questions : 10×2

(a) Determine whether the set {1 – x, 1 + x, 1 – 3x} is linearly dependent on $(-\infty, \infty)$.

(b) Find $\frac{1}{D^2+4}(\sin 2x)$, where $D^2 = \frac{d^2}{dx^2}$.

(c) Find the vector area of the triangle, the position vectors of whose vertices are $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}-\hat{k}$.

[Turn Over]

(d) If $\vec{a}+\vec{b}+\vec{c}=0$, then prove that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

(e) Find the equilibrium point of the system of differential equations $\dot{x}=e^{x-1}-1$ and $\dot{y}=ye^{x}$.

(f) Stae the principle of superposition for homogeneous equation.

(g) If u and v be two independent solutions of the linear equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$, then the wornskian w(u, v) is given by w(u, v) = $Ae^{-\int Pdx}$, where A is a constant.

(h) Show that the three vectors $4\hat{i}+2\hat{j}+\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$ and $8\hat{i}+7\hat{k}$ are coplanar.

(i) Find the directional derivative of $\phi=x^2yz+4xz^2$ at (1, –2, –1) in the direction $2\hat{i}-\hat{j}-2\hat{k}$.

(j) Verify that x = 0 is a singular point of the differential equation

$$2x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}-(x+1)w=0.$$

(k) Find constants a, b and c so that

$$\vec{F}=(x+2y+az)\hat{i}-(bx-3y-z)$$

$\hat{j}+(4x+cy+2z)\hat{k}$ is irrotational.

(l) A particle moves along the curve x = $2t^2$, y = t^2 – 4t, z = 3t – 5, where t is the time. Find the components of its velocity and acceleration at time t = 1 in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

(m) State Lipschitz condition.

(n) Define Euler-cauchy type of equation.

(o) If $\frac{d\vec{a}}{dt}=\vec{r}\times\vec{a}$ and $\frac{d\vec{b}}{dt}=\vec{r}\times\vec{b}$, then show that

$$\frac{d}{dt}[\vec{a}\times\vec{b}]=\vec{r}\times\vec{a}\times\vec{b}$$

where $\vec{r}$ is a constant vector and $\vec{a}$ and $\vec{b}$ are the vector function of a scalar variable t...

2. Answer any four questions : 4×5

(a)

Solve the differential equation $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$ by the method of variation parameter.

[Turn Over]

(b) Solve : $\dfrac{dx}{y-zx}=\dfrac{dy}{x+yz}=\dfrac{dz}{x^2+y^2}$

(c) Solve : $\dfrac{dx}{dt}-7x+y=0$; $\dfrac{dy}{dt}-2x-5y=0$

(d) Show that the volume of the parallelopiped, whose edges are represented by $3\hat{i}+2\hat{j}-4\hat{k}$, $3\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ is 49 cubic units.

(e) (i) Evaluate $\displaystyle\int_{(0,0)}^{(2,1)} (10x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4 - 6xy^3 = 4y^2$.

(ii) Show that $\vec{\nabla}r^n = nr^{n-2}\vec{r}$.

(f) (i) If W be the wronskian of the functions 1, x, x^2, ..., x^{n-1} for $n > 1$, then show that W = 0! 1! 2! ..... (n – 1)!

(ii) Obtain the differential equation of all circles, each of which touches the axis of x at the origin.

3. Answer any two questions : 2×10

(a) Find the series solution of the equation $(x^2 + 1)y_2 + xy_1 - xy = 0$ about $x = 0$.

(b) (i) If $\vec{r}=(a\cos t)\hat{i}+(a\sin t)\hat{j}+(at\tan\alpha)\hat{k}$, then prove that $\left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right]=a^3\tan\alpha$. 5

(ii) Prove that $\displaystyle\int_1^2 \vec{r}\times\frac{d^2\vec{r}}{dt^2}dt=-14\hat{i}+75\hat{j}-15\hat{k}$, where $\vec{r}(t)=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$. 5

(c) (i) If $\vec{F}=4xz\hat{i}-y^2\hat{j}+yz\hat{k}$, evaluate $\displaystyle\iint_S \vec{F}.\hat{n}ds$ where S is the surface of the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1. 5

(ii) Given the space curve $x = t$, $y = t^2$, $z=\frac{2}{3}t^3$, find the curvature k and the torsion Υ. 5

(d) Show that the model represented by $\dfrac{dx}{dt}=x(4-x-y)$; $\dfrac{dy}{dt}-y(15-5x-3y)$, $x\geq 0,\ y\geq 0$ has a position of equilibrium and this position is stable.

[Turn Over]

Group Theory - I

Unit - I

1. Answer any *two* questions : 2×2=4

(a) Prove that (Z, +) is the semigroup.

(b) If each element, except the identity, of a group be of order 2, prove that the group is abelian.

(c) Define symmetric group S_3. What is the identity element of this group.

2. Answer any *one* question : 5×1=5

(a) In a group $(G, \circ)$ in which $(a \circ b)^3 = a^3 \circ b^3$ and $(a \circ b)^5 = a^5 \circ b^5$ for all $a, b \in G$, prove that the group is abelian.

(b) Prove that the set D of all odd integers forms a commutative group with respect to * defined by a * b = a + b − 1 for a, b $\in D$.

Unit - II

3. Answer any *two* questions : 2×2=4

(a) Find the Cyclic subgroups of the group $(s, \circ)$, where $S = \{1, i, -1, -i\}$.

(b) Give an example to show that the union of two subgroups of a group may not be a sub group.

(c) Let (G, o) be a group and KCHCG. If (H, o) be a subgroup of (G, o) and (K, o) be a subgroup of (H, o), prove that (K, o) is a subgroup of (G, o).

4. Answer any *two* questions : 2×5=10

(a) If *H* and *K* happens to be a subgroup of G, then prove that $\circ(HK) = \dfrac{\circ(H)\cdot\circ(K)}{\circ(H \cap K)}$.

(b) Let G be a group in which $(ab)^3 = a^3b^3$ for all a, $b \in G$. Show that $H = \{x^6 : x \in G\}$ is a subgroup of G.

[Turn Over]

(c) Let (G, o) be a group and H be a non-empty subset of G. A relation ρ defined on G by "$a\rho b$ if and only if $a \circ b^{-1} \in H$ for a, $b \in G$, is an equivalence relation on G. Prove that (H, o) is a subgroup of (G, o).

Unit - III

5. Answer any *two* questions : 2×2=4

(a) Let G be a group and H be a subgroup of G. Let $a \in G - H$. The prove that $aH \cap H = \phi$.

(b) G is a cyclic group of order 30 generated by a. Find the order of the cyclic subgroup generated by a^{18}.

(c) Find the inverse of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$$

6. Answer any *one* question : 1×10=10

(a) (i) Prove that every group of prime order is cyclic.

(ii) A cyclic group of finite order n has one and only one subgroup of order d for every positive divisor and of n. 5

(b) (i) Let G = <a> be a cyclic group of order n. Prove that every subgroup H of G is of the form <a^m> where m is a divisor of n. 5

(ii) State and prove Fermat's Little theorem. 5

Unit - IV

7. Answer any *two* questions : 2×2=4

(a) Let G_1, G_2 be two groups and $Z(G_1)$, $Z(G_2)$ be their respective centres. Prove that $Z(G_1) \times Z(G_2)$ is a centre of the group $G_1 \times G_2$.

(b) Find the order of the element $\frac{2}{3} + Z$ in the quotient group Q/Z.

(c) Let H be a subgroup of a group G and [G : H] = 2. Show that H is a normal subgroup of G.

8. Answer any *one* question : 1×10=10

(a) (i) Show that subgroup H of a group G is normal if $aHa^{-1} = H$ for every a in G. 5

[Turn Over]

(ii) Let H be a subgroup of a group G and [G : H] = 2. Then show that H is normal in G 5

(b) Let a, $b \in \mathbb{R}$ and a mapping $T_{a,b} : \mathbb{R} \to \mathbb{R}$ is defined by $T_{ab}(x) = ax + b$, $x \in R$. Let $G = \{T_{ab} : a \neq 0\}$. Prove that (G, o) is a group where o is the composition of mappings. Let $H = \{T_{ab} : a = 1\}$. Prove that H is a normal subgroup of G.

Unit - V

9. Answer any *two* questions : 2×2=4

(a) Let G = (Z, +) and a mapping $\phi : G \to G$ be defined by $\phi(x) = x + 1$, $x \in G$. Examine if ϕ is a homomorphism.

(b) Show that the groups (Q, +) and (R, +) are not isomorphic.

(c) Let (G, o) be a group and a mapping $\phi : G \to G$ is defined by $\phi(x) = x^2, x \in G$. Prove that ϕ is homomorphism if G is commutative.

10. Answer any *one* question : 1×5=5

(a) Let $G = S_3$, $G' = (\{1, -1\}, \bullet)$ and let $\varphi : G \to G'$ be defined by

$\varphi(\alpha) = 1$ if α be an even permutation in S_3

$= -1$ if α be an odd permutation in S_3.

Examine if φ is a homomorphism.

(b) Let $H \subset K \subset G$ and H is normal in K, K is normal in G and also H is normal in G. Show that K/H is normal in G/H and $\frac{G/H}{K/H} \simeq G/K$.

Theory of Real Function and Introduction

Unit - I

1. Answer any *three* questions : 3×2=6

(a) Given an example of jump discontinuity of a function.

(b) Find the points of discontinuity of $f(x) = \frac{1}{1 + \frac{1}{x}}$.

[Turn Over]

(c) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in R$.

(d) If $\lim_{x \to a} |f(x)| = 0$ then show that $\lim_{x \to a} f(x) = 0$.

(e) State intermediate value theorem.

2. Answer any *one* question : 1×5=5

(a) If f(x) and g(x) are two real valued functions of x defined on an interval I such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then prove that $\lim_{x \to a} \{f(x).g(x)\} = lm$, $a \in I$.

(b) Discuss the continuety of the function

$$f(x) = \begin{cases} 2, & x^2 > 4 \\ 3, & x^2 = 4 \\ 0, & x^2 < 4 \end{cases}$$

State the type of discontinuity if f(x) is discontinuous any-where.

3. Answer any one question : [1×10 = 10]

(a) (i) If f(x) = x and g(x) = sin x, show that both f and g are uniformly continuous on R, but that their product fg is not uniformly continuous on R. 5

(ii) Show that

$$\lim_{n \to \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} \right) = 1$$ 5

(b) (i) Prove that if f be continuous on a closed interval, then it assumes its least upper bound and its greatest lower bound in that interval. 6

(ii) Prove that if a function f is uniformly continuous in a certain interval I, then it is necessarily continuous on I. 4

Unit - II [Marks : 14]

4. Answer any *two* questions : 2×2=4

(a) Show that $x > \sin x$ for $0 < x < \frac{\pi}{2}$.

(b) What is the geometrical interpretation of the Rolle's theorem.

[Turn Over]

(c) Find the point of relative extrema of the following function on the specified domain :

$f(x) = 1 - (x-1)^{2/3}$ for $0 \le x \le 2$.

5. Answer any two questions : [2×5 = 10]

(a) State and prove Rolle's theorem.

(b) Verify Lagrange's Mean value theorrem for the function $f(x) = x(x-1)(x-3)$ in $[0, 4]$.

(c) In the mean value theorem $f(a+h) = f(a) + h \cdot f'(a+\theta h)$ if $a = 1$, $h = 3$ and $f(x) = \sqrt{x}$, find θ.

Unit - III [Marks : 14]

6. Answer any *two* questions : 2×2=4

(a) Show that the maximum value of $(x)^{\frac{1}{x}}$ is $(e)^{\frac{1}{e}}$.

(b) Examine the function $f(x) = 4 - 3(x-2)^{2/3}$ for maxima and minima at $x = 2$.

(c) State Maclaetrin's theorem with cauchy's form of remainder.

7. Answer any one question : [1×10 = 10]

(a) (i) Find the minimum and maximum value of

$$f(x) = 3x + \frac{2}{3x} \text{ for all } x \in R - \{0\}.$$

(ii) Assuming the validity of expansion, show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \ldots..$$

(b) (i) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

(ii) Show that $R_n \to 0$ as $n \to \infty$ for the expansions of $(1+x)^m$ in a given range of validity, where R_n is the remainder after *n* terms.

Unit - IV [Marks : 11]

8. Answer any *three* questions : 3×2=6

(a) Define dense set.

(b) Let (X, d) be a metric space. Then show that diameter $\delta(A) = 0$ iff $A \subset X$ is a singleton set.

(c) Define open ball. Give an example.

[Turn Over]

(d) Give two examples of separable metric spaces.

(e) Let (X, d) be a metric space. Then prove that

(i) the null set ϕ is closed,

(ii) X is closed.

9. Answer any one question : [1×5 = 5]

(a) Prove that every non-empty open set in the real line is the union of a countable collection of mutually disjoint open intervals.

(b) If d is a metric on a set X, then proved that d_1 and d_2 are metric where

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

$$d_2(x, y) = \min\{1, d(x, y)\} \text{ for } x, y \in x.$$
