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2019

B.Sc.

3rd Semester Examination

MATHEMATICS (Honours)

Paper - C 5-T

Theory of Real Functions and Introduction to Metric Space

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Unit - 1 [Marks : 21]

1. Answer any *three* questions : $2\times3 = 6$

(a) Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function in $[0, 1]$ prove that there exist a point c in $[0, 1]$ such that $f(c) = c$.

(b) State Cauchy's criteria for the existence of $\lim_{x\to\infty} f(x)$.

[Turn Over]

9/50-2500

(c) Give example of functions f and g which are not continuous at a point $c \in R$ but the sum f + g is continuous at c.

(d) What do you mean by removable discontinuity of a function at an interior point of an interval?

(e) Let $f(x)=\begin{cases}2, x \in Q \\ 0, x \in \mathbb{R}-Q\end{cases}$

Prove that f is discontinuous at every point c in $\mathbb{R}$.

2. Answer any *one* question : 5×1 = 5

(a) Let $D \subset R$, and f and g be functions on D to R and $g(x) \neq 0$ for all $x \in D$. Let $c \in D'$ and $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m \neq 0$. Then prove that $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l}{m}$. 5

(b) A function $f : \mathbb{R}$ is defined by $f(x)=\begin{cases}x, x \in Q \\ 0, x \in \mathbb{R}-Q\end{cases}$ then show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in $\mathbb{R}$. 5

3. Answer any *one* question : 10×1=10

(a) (i) Let A, $B \subset \mathbb{R}$ and $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ be functions s.t $f(A) \subset B$. Let $C \in A$ and f is continuous at c and g is continuous at $f(c) \in B$. Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous at C. 5

(ii) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be functions s.t $f(x) \geq 0 \ \forall x \in D$ and f is continuous on D. Then $\sqrt{f}$ is continuous on D. Hence prove that $h(x) = \sqrt{x^3+3}, x \in R$ is continuous on $\mathbb{R}$. 3+2

(b) (i) Let I be a bounded interval and $f : I \to R$ be uniformly continuous on I. Then prove that f is bounded on I. Show that $\cos\frac{1}{x}$ is not uniformly continuous. 4+2

(ii) Let $f : R \to R$ be strictly increasing and continuous and let S = f(R). Then $f^{-1} : S \to R$ is also strictly increasing and continuous. 4

[Turn Over]

Unit 2 [Marks : 14]

4. Answer any *two* questions : 2×2=4

(a) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

(b) Let $f : R \to R$ is defined by f(x) = |x| + |x − 1|, $x \in R$. Find the derived function f' and specify the domain of f'.

(c) Verify Rolle's theorem for $f(x) = \sin\left(\frac{1}{x}\right)$ on $\left[\frac{1}{3\pi}, \frac{1}{2\pi}\right]$.

5. Answer any *two* questions : 5×2=10

(a) Use MVT to prove the inequality $\frac{x}{\sqrt{1-x^2}} \le \sin^{-1} x < x$ if $0 \le x < 1$. When does the equality hold. 4+1

(b) A function f is twice differentiable on [a, b] and f(a) = f(b) = 0 and f(c) < 0 for some c in (a, b). Prove that there is at least one point ξ in (a, b) for which $f''(\xi) > 0$. 5

(c) When a function is said to be differentiable at a point ? Let I be an interval and $c \in I$. Let the functions $f : I \to R$ and $g : I \to R$ be differentiable at c. Then prove that f.g is differentiable at c and

$(fg)'(c) = f'(c)g(c) + g'(c)f(c)$. 1+4

Unit 3 [Marks : 14]

6. Answer any *two* questions : 2×2=4

(a) State Taylor's theorem with Cauchy's form of remainder.

(b) Find the local extremum points of the function

$$f(x) = \frac{x^2}{(1-x)^3}.$$

(c) A function f is differentiable on [0, 2] and f(0) = 0, f(1) = 2, f(2) = 1. Prove that $f'(c) = 0$ for some c in (0, 2).

7. Answer any *one* question : 10×1=10

(a) (i) State and prove the Maclaurin's theorem with Cauchy's form of remainder.

[Turn Over]

(ii) Use Taylor's theorem to prove that

$$1+\frac{x}{2}-\frac{x^3}{8}<\sqrt{1+x}<1+\frac{x}{2},\ \text{if } x>0.\quad 6+4$$

(b) (i) If $f'(x)=(x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maxima nor a minima at 'a' and f has a minimum at 'b'. 5

(ii) Let $f:I\rightarrow R$ be such that f has a local extrema at an interior point c of I. If $f'(c)$ exists then prove that $f'(c)=0$. 5

Unit 4 [Marks : 11]

8. Answer any three questions : 2×3=6

(a) Prove that in any discrete metric space, are the sets are closed.

(b) Let, $11^{-\wedge}d$ be a metric space. Then prove that $\forall A, B\in M$, $ACB \Rightarrow S(A)\leq S(B)$ where S(X) denote the diameter of a set X.

(c) Let (M, d) be a metric space. Then prove that $(M, \sqrt{d})$ is also a metric space.

(d) Let $X=\mathbb{N}$, the set of natural number, and d ye defined by $d(m, n)=\left|\frac{1}{m}-\frac{1}{n}\right|$, m, $n\in N$, then prove that (X, d) is a discrete metric space.

(e) Let M be a non-empty set for x, $y\in M$, and $d(x, y)=\begin{cases}1, & x\neq y\\ 0, & x=y\end{cases}$ then prove that (M, d) is a metric space.

9. Answer any *one* question : 1×5=5

(a) State and prove that Hausdorff property.

(b) Let $\{A_n\}$ be a sequence of open sets in R_u such that each A_n is dense in $\mathbb{R}_u$. Prove that $\bigcap_{n=1}^{\infty} A_n$ is dense in $\mathbb{R}_u$.