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2018

2nd Semester

MATHEMATICS

PAPER—C4T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Differential Equations and Vector Calculus

Unit-I

[Marks : 22]

1. Answer any *one* question : 1×2

(a) Let ϕ be a solution for $0 < x < \alpha$ of the Euler equation $x^2y'' + axy' + by = 0$ where a, b are constants. Let $\psi(t) = \varphi(e^t)$, then show that ψ satisfies the equation

$$\frac{d^2\psi}{dt^2} + (a-1)\frac{d\psi}{dt} + b\psi = 0 .$$

(Turn Over)

(b) Test wheather the solution e^x, e^{2x}, e^{3x} are linearly independent or not.

2. Answer any *two* questions : 2×5

(a) Knowing that y = x is a solution of the equation

$$x^2 \frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0 \ (x \neq 0)$$

reduce the equation

$$x^2 \frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = x^3 \ (x \neq 0)$$

to a differential equation of first order and first degree and find its complete primitive.

(b) Solve the differential equation :

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \cos x .$$

(c) Solve the equation

$$\frac{d^2y}{dx^2} + a^2y = \tan ax$$

by the method of variation of parameters.

3. Answer any *one* question : 10×1

(a) (i) Solve the differential equation

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x$$

by the method of undetermined co-efficients. 5

(ii) State the sufficient condition for existence and uniqueness of the solution of the differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.

Show that $\frac{dy}{dx} = \frac{1}{y}$, $y(0) = 0$ has more than one solution and indicate the possible reason.

2+2+1

(b) (i) Let a_1, a_2 are continuous functions on [a, b] and ϕ_1, ϕ_2 be the two independent solutions of $y''(x) + a_1(x)\, y'(x) + a_2(x)y(x) = 0$ on some interval [a, b]. Let x_0 be any point in [a, b]. Then show that

$$W(\phi_1, \phi_2)(x) = \exp\left\{-\int_{x_0}^{x} a_1(t)dt\right\} W(\phi_1, \phi_2)(x_0),$$

$$\forall x \in [a, b]$$

where $W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$. 5

(ii) Solve the differential equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}.$$ 5

Unit-II

[Marks : 13]

4. Answer any *four* questions : 2×4

(a) Solve the equations $\frac{dx}{dt}=-wy$ and $\frac{dy}{dt}=wx$ and show that the point (x, y) lies on a circle.

(b) Solve the equation

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}.$$

(c) Find the complementary function for the system

$$(D+3)x+Dy=\cos t$$

$$(D-1)x+y=\sin t$$

where $D\equiv\frac{d}{dt}$.

(d) Solve : $\frac{yzdx}{y-z}=\frac{zxdy}{z-x}=\frac{xydz}{x-y}$.

(e) Show that the solution of the differential equations $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 3x$ satisfies the relation $3x + y = ke^{3t}$ where k is a real constant.

(f) If $\frac{dy_1}{dx} = 3y_1 + 4y_2$ and $\frac{dy_2}{dt} = 4y_1 + 3y_2$ then find the value of $y_1(x)$.

5. Answer any *one* question : 5×1

(a) Find the fundamental matrix and the complementary solution of the homogenious linear system of differential equations

$$\frac{dx}{dt} = 3x + y \quad \text{and} \quad \frac{dy}{dt} = x + 3y.$$ 5

(b) (i) Solve the equation
$$(x^2 + y^2 + z^2)dx - 2xydy - 2xzdz = 0.$$

(ii) Find f(y) such that the total differential

$$\frac{yz + z}{x}dx - zdy + f(y)dz = 0$$

is integrable. Hence solve it. $2\frac{1}{2} + 2\frac{1}{2}$

Unit-III

[Marks : 9]

6. Answer any *two* questions : 2×2

(a) Consider the set of non-linear differential equations

$$\frac{dx}{dt} = x - xy\ ; \quad \frac{dy}{dt} = -y + xy\ .$$

Find the equilibrium points of the system of equations.

(b) Show that x = 0 is a ordinary point and x = 1 is a regular singular point of the ODE

$$x(x-1)\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} + 2x(x-1)y = 0\ .$$

(c) What do you mean by stable and constable critical points.

7. Answer any *one* question : 5×1

(a) Find the phase curve of the system of dynamical equations $\dot{x} = -x - 2y$ and $\dot{y} = 2x - y$. Also show that the system is stable.

(b) Find the power series solution of the equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$$

in power of x about the origin.

Unit-IV

[Marks : 16]

8. Answer any *three* questions : 3×2

(a) Show that the vector

$$\vec{F} = (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k}$$

is irrotational.

(b) Test the continuity of the vector function

$$\vec{f}(t) = |t|\hat{i} - \sin t\,\hat{j} + (1 + \cos t)\hat{k} \quad \text{at } t = 0.$$

(c) If $\vec{A} = \left(3x^2 + 6y\right)\hat{i} - 14yz\,\hat{j} + 20z^2x\,\hat{k}$, evaluate $\int_c \vec{A}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path c : $x = t$, $y = t^2$, $z = t^3$.

(d) Find the unit vector in the direction of the tangent at any point on the curve given by

$$\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + bt\,\hat{k}.$$

(e) Show that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, $\vec{b}\times\left(\vec{c}\times\vec{a}\right)$, $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar.

9. Answer any *one* question : 1×10

(a) (i) If $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$, where $\vec{a}$, $\vec{b}$, n are constants, then prove that $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = \vec{0}$ and $\vec{r} \times \frac{d\vec{r}}{dt} = n\left(\vec{a} \times \vec{b}\right)$. 5

(ii) Derive the volume of a tetrahedron whose co-ordinates of vertices are given. Use it to calculate the volume of the tetrahedron whose vertices are A(2, −1, −3), B(4, 1, 3), C(3, 3, −1) and D(1, 4, 2). 5

(b) (i) Prove that $\left[\left(\vec{\alpha} \times \vec{\beta}\right).\left(\vec{\beta} \times \vec{\gamma}\right).\left(\vec{\gamma} \times \vec{\alpha}\right)\right] = \left[\vec{\alpha}.\vec{\beta}.\vec{\gamma}\right]^2$, where [.] denotes the scalar triple product. 5

(ii) Find $\hat{t}$, $\hat{n}$ for the curve given by

$\vec{r} = \left(e^t \cos t,\ e^t \sin t,\ e^t\right)$ at t = 0. 5