2018

2nd Semester

MATHEMATICS

PAPER—C3T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

(Real Analysis)

Unit-I

(Real Number System and Sets in $\mathbb{R}$)

[Marks : 24]

1. Answer any *two* questions : 2×2

(a) Prove that $S = \{x \in R \; ; \; \sin x \neq 0\}$ is an open set.

(b) Define limit point of set. Prove that a finite set cannot have any limit point.

(c) State Heine-Borel Theorem. Give an example of open cover of the set $S = (0, 1)$.

2. Answer any *two* questions : 5×2

(a) Prove that the set of all upper bounds of a bounded above set admits of a smallest member. 5

(b) Define compact set. If K be a compact set in R, prove that every infinite subset of K has a limit point in K. 1+4

(c) Define derived set of a set. If A' denotes the derived set of A then prove that $(X \cup Y)' = X' \cup Y'$. 1+4

3. Answer any *one* question : 1×10

(a) (i) If $x, y \in \mathbb{R}$ such that $x < y$, then show that there exists a rational number r where $x < r < y$. 5

(ii) Prove that if a set A is open then its complement A^c is closed. 5

(b) (i) Define Interior of a set S (Int S). Show that Int S is an open set. Also show that it is the largest open set contained in S. 1+2+2

(ii) Define limit point of a set. If $S = \left\{\frac{1}{n} : n \in N\right\}$, prove that 0 is the only limit point of S. 1+4

Unit-II

(Real Sequence)

[Marks : 18]

4. Answer any *four* questions : 4×2

(a) Give an example of an increasing sequence converging to the limit 2.

(b) Prove or disprove : product of a divergent sequence and a null sequence is a null sequence.

(c) *Is the sequence* $\{(-2)^n\}$ *monotonic — Justify your* answer.

(d) Give example of divergent sequences $\{x_n\}$ and $\{y_n\}$ such that the sequence $\{x_n y_n\}$ is convergent.

(e) Give an example of a sequence $\{x_n\}$ such that

$$\inf x_n \langle \liminf x_n \langle \limsup x_n \langle \sup x_n .$$

(f) Prove that a sequence diverging to $\propto$ is unbounded above but bounded below.

5. Answer any *one* question : 10×1

(a) (i) For a sequence $\{x_n\}$, if $\lim\limits_{n\to\infty} x_n = l$, prove that

$\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \cdots + x_n}{n} = l$. Hence prove that for a

sequence $\{x_n\}$, if $\lim\limits_{n\to\infty} x_n = l$, where $x_n > 0$, $\forall\, n \in N$,

prove that $\lim\limits_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = l$. 4+1

(ii) Define Cauchy sequence. Prove that the sequence

$\{x_n\}$ where $x_1 = 0$, $x_2 = 1$ and $x_{n+2} = \dfrac{1}{2}(x_{n+1} + x_n)$

for all $n \geq 1$ is a Cauchy sequence. 1+4

(b) (i) Define limit of a sequence. Show that a convergent sequence cannot converge to more than one limit. 1+4

(ii) If $\{u_n\}_n$ be a monotone bounded sequence, prove that exactly one of l.u.b and g.l.b. of $\{u_n\}n$ does not belong to $\{u_n\}_n$. 5

Unit-III

(Infinite Series)

[Marks : 18]

6. Answer any *four* questions : 4×2

(a) Is the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\frac{1}{2}}$ convergent. Justify.

(b) Prove that an absolutely convergent series is convergent.

(c) State Leibnitz's Test of convergence for an alternating series. When is a series said to converge conditionally ?

(d) Using Cauchy's criterion prove that the series $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$ diverges.

(e) Prove that a necessary condition for the convergence of a series $\sum_{n=1}^{\infty} x_n$ is $\lim_{n\to\infty} x_n = 0$.

(f) Test for convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \geq 1$.

7. Answer any *two* questions : 2×5

(a) If $\{u_n\}_n$ is a strictly decreasing sequence of positive real numbers tending to zero, show that the series

$$u_1 - \frac{1}{2}(u_1 + u_2) + \frac{1}{3}(u_1 + u_2 + u_3) - \ldots$$

$$\ldots + \frac{(-1)^{n-1}}{n}(u_1 + u_2 + \cdots + u_n) + \ldots$$ is convergent.

5

(b) State and prove Cauchy's root test for convergence of a series of positive terms. 1+4

(c) Applying Cauchy's Integral test, show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. 2+3
