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PG (NEW) CBCS

M.Sc. Semester-I Examination, 2019

MATHEMATICS

PAPER: MTM-103

(ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS)

Full Marks: 40 **Time: 2 Hours**

1. Answer any four questions of the following: $4 \times 2 = 8$

a) Write down the hypergeometric series represented by F(a, b, c, z). Prove that $F(1, b, b, z) = \frac{1}{1-z}$.

b) Show that $J_n(z)$ is an odd function of z if n is odd.

c) Find all the singularities of the following differential equation and then classify them:

$$(z - z^2)w'' + (1 - 5z)w' - 4w = 0.$$

d) Define orthogonal functions associated with Strum-Liouville problem.

e) Under suitable transformation to be considered by you, prove that Legendre differential equation can be reduced to hypergeometric equation.

f) Let $P_n(z)$ be the Legendre's polynomial of degree n and $p_{m+1}(0) = -\frac{m}{m+1} p_{m-1}(0), m = 1, 2, 3, ...$ If $P_n(0) = -\frac{5}{16}$, then find the value of $\int_{-1}^{1} p_n^2(z)dz$.

g) What do you meant by fundamental matrix of system of linear homogeneous differential equation?

h) Show that $\int_{-1}^{1} p_n(z)dz = \begin{cases} 0, & n \neq 0 \\ 2, & n = 0 \end{cases}$ where the symbol is the usual meaning.

2. Answer any four questions of the following. $4 \times 4 = 16$

a) Using Green's functions method, solve the following differential equation $y''(x) = 0$, subject to boundary conditions $y(0) = y'(1), y'(0) = y(1)$.

b) Establish the generating function for Bessel's function $J_n(z)$. Use it, prove the following: $zJ_n'(z) = zJ_{n-1}(z) - nJ_n(z)$.

c) If $z > 1$, then prove that $P_n(z) < P_{n-1}(z)$.

d) Deduce the integral formula for hypergeometric function.

e) Show that $1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \ldots + (2n+1)P_n(z) = \frac{d}{dz}[P_{n+1}(z) + P_n(z)]$ where $P_n(z)$ denotes the Legendre's Polynomial of degree n.

(Turn Over)

f) If the vector function $\phi_1, \phi_2, \ldots, \phi_n$ defined as follows

$$\phi_1 = \begin{bmatrix} \phi_{11} \\ \cdot \\ \phi_{n1} \end{bmatrix}, \phi_2 = \begin{bmatrix} \phi_{21} \\ \cdot \\ \phi_{n2} \end{bmatrix}, \ldots, \phi_n = \begin{bmatrix} \phi_{1n} \\ \cdot \\ \phi_{nn} \end{bmatrix}$$

be n solutions of the homogeneous linear differential equation $\frac{dx}{dt} =$ A(t)x(t) in the interval $a \le t \le b$, then these n solutions are linearly independent in $a \le t \le b$ iff Wronskian $W[\phi_1, \phi_2, \ldots, \phi_n] \neq 0$ $\forall t$, on a $\le t \le b$.

g) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_0^2(z) = 1$ and prove that real z, $|J_0(z)| \le 1$, and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \ge 1$.

h) Let $W_1(z)$ and $W_2(z)$ be two solutions of $(1 - z^2)w''(z) - 2zw'(z) + (\sec z)w = 0$ with wronskian W(z). If $W_1(0) = 1, W_1'(0) = 0$ and $W\left(\frac{1}{2}\right) = \frac{1}{3}$, then find the value $W_2'(z)$ at z=0.

3. Answer any <u>two</u> questions of the following **$2 \times 8 = 16$**

a) i) All the eigen values of regular SL problem with $r(x) > 0$, are real. **(4)**

ii) Find the general solution of the homogeneous system $\frac{dX}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. **(4)**

b) i) If α *and* β are the roots of the equation $J_n(z) = 0$ then show that

$$\int_0^1 J_n(\alpha z) J_n(\beta z) dz = \begin{cases} 0, & if\ \alpha \neq \beta \\ \frac{1}{2}[J_n'(z)]^2, & if\ \alpha = \beta. \end{cases}$$ **(6)**

ii) Show that $\int_0^{\infty} \frac{J_n(z)}{z} dz = \frac{1}{n}$ $(n \neq 0)$ **(2)**

c) i) Prove that if f(z) is continuous and has continuous derivatives in [-1,1] then f(z) has unique Legendre series expansion is given by $f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$ where P_n's are Legendre polynomials $C_n = \frac{2n+1}{2} \int_{-1}^{1} f(z) P_n(z)$, $n = 1,2,3 \ldots$ **(6)**

ii) Prove that $\frac{d}{dz}[J_0(z)] = -J_1(z)$. **(2)**

d) Find the general solution of the equation $2z(1 - z)\frac{d^2w}{dz^2} + \frac{dw}{dz} + 4w(z) = 0$ by Frobenious method about z=0 and show that the equation has solution which is polynomial in z. **(7+1)**

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