

PG (NEW) CBCS
M.Sc. Semester-I Examination, 2019
MATHEMATICS
PAPER: MTM-103



(ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS)

Full Marks: 40

Time: 2 Hours

1. Answer any four questions of the following:

4 × 2 = 8

- a) Write down the hypergeometric series represented by $F(a, b, c, z)$. Prove that $F(1, b, b, z) = \frac{1}{1-z}$.
- b) Show that $J_n(z)$ is an odd function of z if n is odd.
- c) Find all the singularities of the following differential equation and then classify them:

$$(z - z^2)w'' + (1 - 5z)w' - 4w = 0.$$

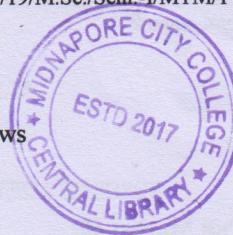
- d) Define orthogonal functions associated with Sturm-Liouville problem.
- e) Under suitable transformation to be considered by you, prove that Legendre differential equation can be reduced to hypergeometric equation.
- f) Let $P_n(z)$ be the Legendre's polynomial of degree n and $p_{m+1}(0) = -\frac{m}{m+1}p_{m-1}(0)$, $m = 1, 2, 3, \dots$. If $P_n(0) = -\frac{5}{16}$, then find the value of $\int_{-1}^1 p_n^2(z) dz$.
- g) What do you mean by fundamental matrix of system of linear homogeneous differential equation?
- h) Show that $\int_{-1}^1 p_n(z) dz = \begin{cases} 0, & n \neq 0 \\ 2, & n = 0 \end{cases}$ where the symbol is the usual meaning.

2. Answer any four questions of the following.

4 × 4 = 16

- a) Using Green's functions method, solve the following differential equation $y''(x) = 0$, subject to boundary conditions $y(0) = y'(1)$, $y'(0) = y(1)$.
- b) Establish the generating function for Bessel's function $J_n(z)$. Use it, prove the following: $zJ'_n(z) = zJ_{n-1}(z) - nJ_n(z)$.
- c) If $z > 1$, then prove that $P_n(z) < P_{n-1}(z)$.
- d) Deduce the integral formula for hypergeometric function.
- e) Show that $1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \dots + (2n+1)P_n(z) = \frac{d}{dz}[P_{n+1}(z) + P_n(z)]$ where $P_n(z)$ denotes the Legendre's Polynomial of degree n .

(Turn Over)



(2)

f) If the vector function $\phi_1, \phi_2, \dots, \phi_n$ defined as follows

$$\phi_1 = \begin{bmatrix} \phi_{11} \\ \cdot \\ \phi_{n1} \end{bmatrix}, \phi_2 = \begin{bmatrix} \phi_{21} \\ \cdot \\ \phi_{n2} \end{bmatrix}, \dots, \phi_n = \begin{bmatrix} \phi_{1n} \\ \cdot \\ \phi_{nn} \end{bmatrix}$$

be n solutions of the homogeneous linear differential equation $\frac{dx}{dt} =$

$A(t)x(t)$ in the interval $a \leq t \leq b$, then these n solutions are linearly independent in $a \leq t \leq b$ iff Wronskian $W[\phi_1, \phi_2, \dots, \phi_n] \neq 0 \forall t$, on $a \leq t \leq b$.

g) Show that $J_0^2(z) + 2 \sum_{n=1}^{\infty} J_0^2(z) = 1$ and prove that real z , $|J_0(z)| \leq 1$, and $|J_n(z)| < \frac{1}{\sqrt{z}}$ for all $n \geq 1$.

h) Let $W_1(z)$ and $W_2(z)$ be two solutions of $(1 - z^2)w''(z) - 2zw'(z) + (\sec z)w = 0$ with wronskian $W(z)$. If $W_1(0) = 1, W_1'(0) = 0$ and $W\left(\frac{1}{2}\right) = \frac{1}{3}$, then find the value $W_2'(z)$ at $z=0$.

3. Answer any two questions of the following

2 × 8 = 16

a) i) All the eigen values of regular SL problem with $r(x) > 0$, are real.

(4)

ii) Find the general solution of the homogeneous system $\frac{dX}{dt} =$

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(4)

b) i) If α and β are the roots of the equation $J_n(z) = 0$ then show that

$$\int_0^1 J_n(\alpha z) J_n(\beta z) dz = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2} [J_n'(z)]^2, & \text{if } \alpha = \beta. \end{cases} \quad (6)$$

ii) Show that $\int_0^{\infty} \frac{J_n(z)}{z} dz = \frac{1}{n}$ ($n \neq 0$) (2)

c) i) Prove that if $f(z)$ is continuous and has continuous derivatives in $[-1, 1]$ then $f(z)$ has unique Legendre series expansion is given by $f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$ where P_n 's are Legendre polynomials

$$C_n = \frac{2n+1}{2} \int_{-1}^1 f(z) P_n(z) dz, \quad n = 1, 2, 3 \dots \quad (6)$$

ii) Prove that $\frac{d}{dz} [J_0(z)] = -J_1(z)$. (2)

d) Find the general solution of the equation $2z(1-z) \frac{d^2 w}{dz^2} + \frac{dw}{dz} + 4w(z) = 0$ by Frobenius method about $z=0$ and show that the equation has solution which is polynomial in z . (7+1)
